“Minimal geometric data” approach to Dirac algebra, spinor groups and field theories

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Abstract

The three first sections contain an updated, not-so-short account of a partly original approach to spinor geometry and field theories introduced by Jadczyk and myself [3, 4, 5]; it is based on an intrisic treatment of 2-spinor geometry in which the needed background structures have not to be assumed, but rather arise naturally from a unique geometric datum: a vector bundle with complex 2-dimensional fibres over a real 4-dimensional manifold. The two following sections deal with Dirac algebra and 4-spinor groups in terms of two spinors, showing various aspects of spinor geometry from a different perspective. The last section examines particle momenta in 2-spinor terms and the bundle structure of 4-spinor space over momentum space.

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Introduction

The precise equivalence between the 4-spinor and 2-spinor settings for electrodynamics was exposed by Jadczyk and myself in [2, 3, 4, 5]. In summary one sees that, from an algebraic point of view, the only notion of a complex 2-dimensional vector space $\boldsymbol{S}$ yields, naturally and without any further assumptions, all the needed algebraic structures through functorial constructions; conversely in a 4-spinor setting, provided one makes the minimum assumptions which are needed in order to formulate the standard physical theory, the 4-spinor space naturally splits (Weyl decomposition) into the direct sum of two 2-dimensional subspaces which are anti-dual to each other. In a sense, which setting one regards as fundamental is then mainly a matter of taste. The 4-spinor setting is closer to standard notations, and some formulas can be written in a more compact way, while the relations among the various objects are somewhat more involved. The 2-spinor setting turns out to give a much more direct formulation, in which all the basic objects and the relations among them naturally set into their places; just from $\boldsymbol{S}$ one authomatically gets *exactly* the needed algebraic structure, nothing more, nothing less: 4-spinor space $\boldsymbol{W}$ with the 'Dirac adjoint' anti-isomorphism, Minkowski space $\boldsymbol{H}$ and Dirac map $\gamma : \boldsymbol{H} \to \mathrm{End}(\boldsymbol{W})$ with the required properties. Further objects which are commonly considered depend on the choice of a gauge of some sort, whose nature is precisely described.

When we consider a vector bundle $\boldsymbol{S} \to \boldsymbol{M}$, where now the fibres are complex 2-dimensional and $\boldsymbol{M}$ is a real 4-dimensional manifold, then we don't have to assign any further background structure in order to formulate a full Einstein-Cartan-Maxwell-Dirac theory. In fact we naturally get a vector bundle $\boldsymbol{H} \to \boldsymbol{M}$ whose fibres are

Minkowski spaces, a 4-spinor bundle $\boldsymbol{W} \to \boldsymbol{M}$ and so on. Any object which is not determined by geometric construction from the unique geometric datum $\boldsymbol{S} \to \boldsymbol{M}$ is a *field* of the theory, namely we consider: the tetrad $\Theta : \mathrm{T}\boldsymbol{M} \to \mathbb{L} \otimes \boldsymbol{H}$, the 2-spinor connection $\digamma$, the electromagnetic and Dirac fields. (Even coupling factors naturally arise as covariantly constant sections of the real line bundle $\mathbb{L}$ of *length units*, which is geometrically constructed from $\boldsymbol{S}$.) The gravitational field is described by the tetrad (which can be seen as a 'square root' of spacetime metric) and by the connection induced by $\digamma$ on $\boldsymbol{H}$, while the remaining part of the spinor connection can be viewed as the electromagnetic potential. A natural Lagrangian density for all these fields is then introduced; the relations between metric and connection and between e.m. potential and e.m. field follow from the (Euler-Lagrange) field equations. All considered, this setting has some original aspects but is not in contrast to the (by now classical) Penrose formalism [12].

In §4 and §5 I'll show how the above said algebraic setting, and in particular the natural splitting of the 4-spinor space into the direct sum of its Weyl subspaces, enables us to examine the structures of the Dirac algebra, the Clifford group and its subgroups from a different perspective.

In §6 I'll show the strict relation existing between the two-spinor setting and the geometry of particle momenta, in particular the bundle structure of $\boldsymbol{W}$ over the space of momenta. These results are a preparation to a 2-spinor formulation of quantum electrodynamics along le lines of a previous paper [6], in which the classical structure underlying electron states is a 2-fibred bundle over spacetime.

1 Two-spinor geometry

In this section we'll see how all the fundamental geometric structures needed for Dirac theory naturally arise through functorial constructions from a two-dimensional complex vector space, with no further assumptions.

1.1 Complex conjugated spaces

If $\boldsymbol{A}$ is a set and $f : \boldsymbol{A} \to \mathbb{C}$ is any map, then $\bar{f} : \boldsymbol{A} \to \mathbb{C} : a \mapsto \overline{f(a)}$ is the conjugated map. Let $\boldsymbol{V}$ be a complex vector space of finite-dimension n; its *dual* space $\boldsymbol{V}^\star$ and *antidual* space $\boldsymbol{V}^{\overline{\star}}$ are defined to be the n-dimensional complex vector spaces of all maps $\boldsymbol{V} \to \mathbb{C}$ which are respectively linear and antilinear. One then has the distinguished anti-isomorphism $\boldsymbol{V}^\star \to \boldsymbol{V}^{\overline{\star}} : \lambda \mapsto \bar{\lambda}$.

Set now $\overline{\boldsymbol{V}} := \boldsymbol{V}^{\star\overline{\star}}$, and call this the *conjugate space* of $\boldsymbol{V}$. One has the natural isomorphisms

$$\boldsymbol{V} \cong \boldsymbol{V}^{\star\star} \cong \boldsymbol{V}^{\overline{\star}\overline{\star}}\,, \quad \overline{\boldsymbol{V}} := \boldsymbol{V}^{\star\overline{\star}} \cong \boldsymbol{V}^{\overline{\star}\star}\,.$$

Summarizing, one one gets the four distinct spaces

$$\boldsymbol{V} \leftrightarrow \overline{\boldsymbol{V}}\,, \quad \boldsymbol{V}^\star \leftrightarrow \boldsymbol{V}^{\overline{\star}}\,,$$

where the arrows indicate the conjugation anti-isomorphisms.

Accordingly, coordinate expressions have four types of indices. Let (b_A), $1 \leq A \leq n$, be a basis of $\boldsymbol{V}$ and (b^A) its dual basis of $\boldsymbol{V}^\star$. The corresponding indices in the conjugate spaces are distinguished by a dot, namely one writes

$$\bar{\mathsf{b}}_{A^{\centerdot}} := \overline{\mathsf{b}_A}\,, \quad \bar{\mathsf{b}}^{A^{\centerdot}} := \overline{\mathsf{b}^A}\,,$$

so that $\{\bar{\mathsf{b}}_{A^\cdot}\}$ is a basis of $\overline{\boldsymbol{V}}$ and $\{\bar{\mathsf{b}}^{A^\cdot}\}$ its dual basis of $\overline{\boldsymbol{V}}^\star$. For $v \in \boldsymbol{V}$ and $\lambda \in \boldsymbol{V}^\star$ one has

$$v = v^A\,\mathsf{b}_A\ , \quad \bar{v} = \bar{v}^{A^\cdot}\bar{\mathsf{b}}_{A^\cdot}\ ,$$
$$\lambda = \lambda_A\,\mathsf{b}^A\ , \quad \bar{\lambda} = \bar{\lambda}_{A^\cdot}\bar{\mathsf{b}}^{A^\cdot}\ ,$$

where $\bar{v}^{A^\cdot} = \overline{v^A}$, $\bar{\lambda}_{A^\cdot} := \overline{\lambda_A}$ and Einstein summation convention is used.

The conjugation morphism can be extended to tensors of any rank and type; if τ is a tensor then all indices of $\bar{\tau}$ are of reversed (dotted/non-dotted) type; observe that dotted indices cannot be contracted with non-dotted indices. In particular if $K \in \mathrm{Aut}(\boldsymbol{V}) \subset \boldsymbol{V}\otimes\boldsymbol{V}^\star$ then $\bar{K} \in \mathrm{Aut}(\overline{\boldsymbol{V}}) \subset \overline{\boldsymbol{V}}\otimes\overline{\boldsymbol{V}}^\star$ is the induced conjugated transformation (under a basis transformation, dotted indices transform with the conjugate matrix).

1.2 Hermitian tensors

The space $\boldsymbol{V}\otimes\overline{\boldsymbol{V}}$ has a natural real linear (complex anti-linear) involution $w \mapsto w^\dagger$, which on decomposable tensors reads

$$(u\otimes\bar{v})^\dagger = v\otimes\bar{u}\ .$$

Hence one has the natural decomposition of $\boldsymbol{V}\otimes\overline{\boldsymbol{V}}$ into the direct sum of the *real* eigenspaces of the involution with eigenvalues ± 1, respectively called the *Hermitian* and *anti-Hermitian* subspaces, namely

$$\boldsymbol{V}\otimes\overline{\boldsymbol{V}} = (\boldsymbol{V}\,\bar{\vee}\,\overline{\boldsymbol{V}}) \oplus \mathrm{i}\,(\boldsymbol{V}\,\bar{\vee}\,\overline{\boldsymbol{V}})\ .$$

In other terms, the Hermitian subspace $\boldsymbol{V}\,\bar{\vee}\,\overline{\boldsymbol{V}}$ is constituted by all $w \in \boldsymbol{V}\otimes\overline{\boldsymbol{V}}$ such that $w^\dagger = w$, while an arbitrary w is uniquely decomposed into the sum of an Hermitian and an anti-Hermitian tensor as

$$w = \tfrac{1}{2}(w + w^\dagger) + \tfrac{1}{2}(w - w^\dagger)\ .$$

In terms of components in any basis, $w = w^{AB^\cdot}\mathsf{b}_A\otimes\bar{\mathsf{b}}_{B^\cdot}$ is Hermitian (anti-Hermitian) iff the matrix $(w^{AB^\cdot})$ of its components is of the same type, namely $\bar{w}^{B^\cdot A} = \pm w^{AB^\cdot}$.

Obviously $\boldsymbol{V}^\star\otimes\overline{\boldsymbol{V}}^\star$ decomposes in the same way, and one has the natural isomorphisms

$$(\boldsymbol{V}\,\bar{\vee}\,\overline{\boldsymbol{V}})^* \cong \boldsymbol{V}^\star\,\bar{\vee}\,\overline{\boldsymbol{V}}^\star\ , \quad (\mathrm{i}\,\boldsymbol{V}\,\bar{\vee}\,\overline{\boldsymbol{V}})^* \cong \mathrm{i}\,\boldsymbol{V}^\star\,\bar{\vee}\,\overline{\boldsymbol{V}}^\star\ ,$$

where * denotes the *real* dual.

A Hermitian 2-form is defined to be a Hermitian tensor $h \in \overline{\boldsymbol{V}}^\star\,\bar{\vee}\,\boldsymbol{V}^\star$. The associated quadratic form $v \mapsto h(v,v)$ is real-valued. The notions of signature and non-degeneracy of Hermitian 2-forms are introduced similarly to the case of real bilinear forms. If h is non-degenerate then it yields the isomorphism $h^\flat : \overline{\boldsymbol{V}} \to \boldsymbol{V}^\star : \bar{v} \mapsto h(\bar{v}, \_)$; its conjugate map is an anti-isomorphism $\overline{\boldsymbol{V}} \to \overline{\boldsymbol{V}}^\star$ which, via composition with the canonical conjugation, can be seen as the isomorphism $\bar{h}^\flat : \boldsymbol{V} \to \overline{\boldsymbol{V}}^\star : v \mapsto h(\_, v)$. The inverse isomorphisms $h^\#$ and $\bar{h}^\#$ are similarly related to a Hermitian tensor $h^{-1} \in \overline{\boldsymbol{V}}\,\bar{\vee}\,\boldsymbol{V}$. One has the coordinate expressions

$$(h^\flat(\bar{v}))_B = h_{A^\cdot B}\bar{v}^{A^\cdot}\ , \qquad (\bar{h}^\flat(v))_{A^\cdot} = h_{A^\cdot B}v^B = \bar{h}_{BA^\cdot}v^B\ ,$$
$$(h^\#(\bar{\lambda}))^B = h^{A^\cdot B}\bar{\lambda}_{A^\cdot}\ , \qquad (\bar{h}^\#(\lambda))^{A^\cdot} = h^{A^\cdot B}\lambda_B = \bar{h}^{BA^\cdot}\lambda_B\ ,$$

where $h^{C^\cdot A}h_{C^\cdot B} = \delta^A{}_B$, $h^{A^\cdot C}h_{B^\cdot C} = \delta^{A^\cdot}{}_{B^\cdot}$.

1.3 Two-spinor space

Let $\boldsymbol{S}$ be a 2-dimensional complex vector space. Then $\wedge^2\boldsymbol{S}$ is a 1-dimensional complex vector space; its dual space $(\wedge^2\boldsymbol{S})^\star$ will be identified with $\wedge^2\boldsymbol{S}^\star$ via the rule[1]

$$\omega(s\wedge s') := \tfrac{1}{2}\omega(s,s')\ ,\quad \forall\,\omega\in\wedge^2\boldsymbol{S}^\star,\ s,s'\in\boldsymbol{S}\ .$$

Any $\omega\in\wedge^2\boldsymbol{S}^\star\setminus\{0\}$ (a 'symplectic' form on $\boldsymbol{S}$) has a unique 'inverse' or 'dual' element ω^{-1}. Denoting by $\omega^\flat:\boldsymbol{S}\to\boldsymbol{S}^\star$ the linear map defined by $\langle\omega^\flat(s),t\rangle := \omega(s,t)$ and by $\omega^\#:\boldsymbol{S}^\star\to\boldsymbol{S}$ the linear map defined by $\langle\mu,\omega^\#(\lambda)\rangle := \omega^{-1}(\lambda,\mu)$, one has

$$\omega^\# = -(\omega^\flat)^{-1}\ .$$

The Hermitian subspace of $(\wedge^2\boldsymbol{S})\otimes(\wedge^2\overline{\boldsymbol{S}})$ is a 1-dimensional real vector space with a distinguished orientation, whose positively oriented semispace

$$\mathbb{L}^2 := [(\wedge^2\boldsymbol{S})\,\bar{\vee}\,(\wedge^2\overline{\boldsymbol{S}})]^+ := \{w\otimes\bar{w},\ w\in\wedge^2\boldsymbol{S}\}$$

has the square root semi-space $\mathbb{L}$, called the space of *length units*.[2]

Next, consider the complex 2-dimensional space

$$\boldsymbol{U} := \mathbb{L}^{-1/2}\otimes\boldsymbol{S}\ .$$

This is our 2-*spinor space*. Observe that the 1-dimensional space

$$\boldsymbol{Q} := \wedge^2\boldsymbol{U} = \mathbb{L}^{-1}\otimes\wedge^2\boldsymbol{S}$$

has a distinguished Hermitian metric, defined as the unity element in

$$\overline{\boldsymbol{Q}}^\star\,\bar{\vee}\,\boldsymbol{Q}^\star \equiv (\wedge^2\overline{\boldsymbol{U}}^\star)\,\bar{\vee}\,(\wedge^2\boldsymbol{U}^\star) = \mathbb{L}^{-2}\otimes(\wedge^2\boldsymbol{S}^\star)\,\bar{\vee}\,(\wedge^2\boldsymbol{S}^\star)\cong\mathbb{R}\ .$$

Hence there is the distinguished set of normalized symplectic forms on $\boldsymbol{U}$, any two of them differing by a phase factor.[3]

Consider an arbitrary basis (ξ_A) of $\boldsymbol{S}$ and its dual basis (x^A) of $\boldsymbol{S}^\star$. This determines the mutually dual bases

$$\mathsf{w} := \varepsilon^{AB}\,\xi_A\wedge\xi_B\ ,\quad \mathsf{w}^{-1} := \varepsilon_{AB}\,\mathsf{x}^A\wedge\mathsf{x}^B\ ,$$

respectively of $\wedge^2\boldsymbol{S}$ and $\wedge^2\boldsymbol{S}^\star$ (here ε^{AB} and ε_{AB} both denote the antisymmetric Ricci matrix), and the basis

$$l := \sqrt{\mathsf{w}\otimes\bar{\mathsf{w}}}\quad\text{of}\quad\mathbb{L}\ .$$

Then one also has the induced mutually dual, *normalized* bases

$$(\zeta_A) := (l^{-1/2}\otimes\xi_A)\ ,\quad (\mathsf{z}^A) := (l^{1/2}\otimes\mathsf{x}^A)$$

[1] Here, $s\wedge s' := \frac{1}{2}(s\otimes s' - s'\otimes s)$. This contraction, defined in such a way to respect usual conventions in two-spinor literature, corresponds to 1/4 standard exterior-algebra contraction.

[2] A *unit space* is defined to be a 1-dimensional real semi-space, namely a positive semi-field associated with the semi-ring $\mathbb{R}^+$ (see [1, 2] for details). The *square root* $\mathbb{U}^{1/2}$ of a unit space $\mathbb{U}$, is defined by the condition that $\mathbb{U}^{1/2}\otimes\mathbb{U}^{1/2}$ be isomorphic to $\mathbb{U}$. More generally, any *rational power* of a unit space is defined up to isomorphism (negative powers correspond to dual spaces). In this article we only use the unit space $\mathbb{L}$ of lengths and its powers; essentially, this means that we take $\hbar = c = 1$.

[3] One says that elements of $\boldsymbol{U}$ and of its tensor algebra are 'conformally invariant', while tensorializing by $\mathbb{L}^r$ one obtains 'conformal densities' of weight r.

of $\boldsymbol{U}$ and $\boldsymbol{U}^\star$, and also

$$\varepsilon := l\otimes \mathsf{w}^{-1} = \varepsilon_{AB}\,\mathsf{z}^A\wedge\mathsf{z}^B \in \boldsymbol{Q}^\star \equiv \wedge^2\boldsymbol{U}^\star \ ,$$
$$\varepsilon^{-1} \equiv l^{-1}\otimes\mathsf{w} = \varepsilon^{AB}\,\zeta_A\wedge\zeta_B \in \boldsymbol{Q}\equiv\wedge^2\boldsymbol{U} \ .$$

Remark. In contrast to the usual 2-spinor formalism, no symplectic form is fixed. The 2-form ε is unique up to a phase factor which depends on the chosen 2-spinor basis, and determines isomorphisms

$$\varepsilon^\flat : \boldsymbol{U}\to\boldsymbol{U}^\star : u\mapsto u^\flat\ ,\quad \langle u^\flat, v\rangle := \varepsilon(u,v) \quad\Rightarrow\quad (u^\flat)_B = \varepsilon_{AB}\,v^A\ ,$$
$$\varepsilon^\# : \boldsymbol{U}^\star\to\boldsymbol{U} : \lambda\mapsto\lambda^\#\ ,\quad \langle \mu,\lambda^\#\rangle := \varepsilon^{-1}(\lambda,\mu) \quad\Rightarrow\quad (\lambda^\#)^B = \varepsilon^{AB}\,\lambda_A\ .$$

If no confusion arises, we'll make the identification $\varepsilon^\#\equiv\varepsilon^{-1}$.

1.4 2-spinors and Minkowski space

Though a normalized element $\varepsilon\in\boldsymbol{Q}^\star$ is unique only up to a phase factor, certain objects which can be expressed through it are natural geometric objects. The first example is the unity element in $\boldsymbol{Q}^\star\otimes\overline{\boldsymbol{Q}}{}^\star$, which can be written as $\varepsilon\otimes\bar\varepsilon$; it can also be seen as a bilinear form g on $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$, given for decomposable elements by

$$g(p\otimes\bar q, r\otimes\bar s) = \varepsilon(p,r)\,\bar\varepsilon(\bar q,\bar s)\ .$$

The fact that any ε is non-degenerate implies that g is non-degenerate too. In a normalized 2-spinor basis (ζ_A) one writes $w = w^{AA\dot{}}\,\zeta_A\otimes\bar\zeta_{A\dot{}}\in\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$, $g_{AA\dot{}BB\dot{}} = \varepsilon_{AB}\,\bar\varepsilon_{A\dot{}B\dot{}}$ and[4]

$$g(w,w) = \varepsilon_{AB}\,\bar\varepsilon_{A\dot{}B\dot{}}\,w^{AA\dot{}}\,w^{BB\dot{}} = 2\,\det w\ .$$

Next, consider the Hermitian subspace

$$\boldsymbol{H} := \boldsymbol{U}\,\bar\vee\,\overline{\boldsymbol{U}}\subset\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\ .$$

This is a 4-dimensional *real* vector space; for any given normalized basis (ζ_A) of $\boldsymbol{U}$ consider, in particular, the *Pauli basis* (τ_λ) of $\boldsymbol{H}$ associated with (ζ_A), namely

$$\tau_\lambda \equiv \tau_\lambda{}^{AA\dot{}}\,\zeta_A\otimes\bar\zeta_{A\dot{}} \equiv \tfrac{1}{\sqrt2}\,\sigma_\lambda{}^{AA\dot{}}\,\zeta_A\otimes\bar\zeta_{A\dot{}}\ ,\qquad \lambda = 0,1,2,3\ ,$$

where $(\sigma_\lambda{}^{AA\dot{}})$ denotes the λ-th Pauli matrix.[5]

The restriction of g to the Hermitian subspace $\boldsymbol{H}$ turns out to be a Lorentz metric with signature $(+,-,-,-)$. Actually, a Pauli basis is readily seen to be orthonormal, namely $g_{\lambda\mu} := g(\tau_\lambda\,,\tau_\mu) = \eta_{\lambda\mu} := 2\,\delta^0_\lambda\delta^0_\mu - \delta_{\lambda\mu}$.

It's not difficult to prove:

Proposition 1.1 *An element $w\in\boldsymbol{U}\otimes\overline{\boldsymbol{U}} = \mathbb{C}\otimes\boldsymbol{H}$ is null, that is $g(w,w)=0$, iff it is a decomposable tensor: $w = u\otimes\bar s$, $u,s\in\boldsymbol{U}$.*

[4] Note how $\det w\equiv\det\big(w^{AA\dot{}}\big)$ is intrinsically defined through ε, even if w is not an endomorphism.

[5] $\sigma_0 := \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\sigma_1 := \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 := \begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}$, $\sigma_3 := \begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

A null element in $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ is also in $\boldsymbol{H}$ iff it is of the form $\pm u\otimes\bar{u}$. Hence the *null cone* $\boldsymbol{N}\subset\boldsymbol{H}$ is constituted exactly by such elements. Note how this fact yields a way of distinguish between time orientations: by convention, one chooses the *future* and *past* null-cones in $\boldsymbol{H}$ to be, respectively,

$$\boldsymbol{N}^+ := \{u\otimes\bar{u},\ u\in\boldsymbol{U}\}\ ,\quad \boldsymbol{N}^- := \{-u\otimes\bar{u},\ u\in\boldsymbol{U}\}\ .$$

Proposition 1.2 *For each g-orthonormal positively oriented basis (e_λ) of $\boldsymbol{H}$, such that e_0 is timelike and future-oriented, there exists a normalized 2-spinor basis (ζ_A) whose associated Pauli basis (τ_λ) coincides with (e_λ).*

Remark. From the above proposition it follows that any future-pointing timelike vector can be written as $u\otimes\bar{u}+v\otimes\bar{v}$, for suitable $u,v\in\boldsymbol{U}$.

1.5 From 2-spinors to 4-spinors

Next observe that an element of $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ can be seen as a linear map $\overline{\boldsymbol{U}}^\star\to\boldsymbol{U}$, while an element of $\overline{\boldsymbol{U}}^\star\otimes\boldsymbol{U}^\star$ can be seen as a linear map $\boldsymbol{U}\to\overline{\boldsymbol{U}}^\star$. Then one defines the linear map

$$\gamma:\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\to\mathrm{End}(\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^\star): y\mapsto\gamma(y):=\sqrt{2}\,\big(y,y^{\flat\star}\big)\ ,$$

$$\text{i.e.}\quad \gamma(y)(u,\chi)=\sqrt{2}\big(y\rfloor\chi\,,u\rfloor y^\flat\big)\ ,$$

where $y^\flat := g^\flat(y)\in\boldsymbol{U}^\star\otimes\overline{\boldsymbol{U}}^\star$ and $y^{\flat\star}\in\overline{\boldsymbol{U}}^\star\otimes\boldsymbol{U}^\star$ is the transposed tensor. In particular for a decomposable $y=p\otimes\bar{q}$ one has

$$\tilde{\gamma}(p\otimes\bar{q})(u,\chi)=\sqrt{2}\big(\langle\chi,\bar{q}\rangle\,p\,,\langle p^\flat,u\rangle\,\bar{q}^\flat\big)\ .$$

Proposition 1.3 *For all $y,y'\in\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ one has*

$$\gamma(y)\circ\gamma(y')+\gamma(y')\circ\gamma(y)=2\,g(y,y')\,1\!\!1\ .$$

PROOF: It is sufficient to check the statement's formula for any couple of null i.e. decomposable elements in $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$. Using the identity

$$\varepsilon(p,q)\,r^\flat+\varepsilon(q,r)\,p^\flat+\varepsilon(r,p)\,q^\flat=0\ ,\quad p,q,r\in\boldsymbol{U}\ ,$$

which is in turn easily checked, a straightforward calculation gives

$$\begin{aligned}&[\gamma(p\otimes\bar{q})\circ\gamma(r\otimes\bar{s})+\gamma(r\otimes\bar{s})\circ\gamma(p\otimes\bar{q})](u+\chi)=\\&\qquad=2\,\varepsilon(p,r)\,\bar{\varepsilon}(\bar{q},\bar{s})\,(u,\chi)=2\,g(p\otimes\bar{q},r\otimes\bar{s})\,(u,\chi)\ .\end{aligned}$$

□

Now one sees that γ is a *Clifford map* relatively to g (see also §4.1); thus one is led to regard

$$\boldsymbol{W}:=\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^\star$$

as the space of Dirac spinors, decomposed into its Weyl subspaces. Actually, the restriction of γ to the Minkowski space $\boldsymbol{H}$ turns out to be a Dirac map.

The 4-dimensional complex vector space $\boldsymbol{W}$ is naturally endowed with a further structure: the obvious anti-isomorphism

$$\boldsymbol{W} \to \boldsymbol{W}^\star : (u,\chi) \mapsto (\bar\chi, \bar u) \ .$$

Namely, if $\psi = (u,\chi) \in \boldsymbol{W}$ then $\bar\psi = (\bar u, \bar\chi) \in \overline{\boldsymbol{W}}$ can be identified with $(\bar\chi,\bar u) \in \boldsymbol{W}^\star$; this is the so-called 'Dirac adjoint' of ψ. This operation can be seen as the "index lowering anti-isomorphism" related to the Hermitian product

$$\mathrm{k} : \boldsymbol{W} \times \boldsymbol{W} \to \mathbb{C} : \Big((u,\chi),(u',\chi')\Big) \mapsto \langle \bar\chi, u'\rangle + \langle \chi', \bar u\rangle \ ,$$

which is obviously non-degenerate; its signature turns out to be $(+\,+\,-\,-)$, as it can be seen in a "Dirac basis" (below).

Let (ζ_A) be a normalized basis of $\boldsymbol{U}$; the *Weyl basis* of $\boldsymbol{W}$ is defined to be the basis (ζ_α), $\alpha = 1,2,3,4$, given by

$$(\zeta_1\,,\zeta_2\,,\zeta_3,\zeta_4) := (\zeta_1\,,\zeta_2\,,-\bar{\mathsf{z}}^1,-\bar{\mathsf{z}}^2) \ .$$

Above, ζ_1 is a simplified notation for $(\zeta_1\,,0)$, and the like. Another important basis is the *Dirac basis* (ζ'_α), $\alpha = 1,2,3,4$, where

$$\zeta'_1 = \tfrac{1}{\sqrt2}(\zeta_1\,,\bar{\mathsf{z}}^1) \equiv \tfrac{1}{\sqrt2}(\zeta_1 - \zeta_3) \ , \qquad \zeta'_2 = \tfrac{1}{\sqrt2}(\zeta_2\,,\bar{\mathsf{z}}^2) \equiv (\zeta_2 - \zeta_4) \ ,$$

$$\zeta'_3 = \tfrac{1}{\sqrt2}(\zeta_1\,,-\bar{\mathsf{z}}^1) \equiv (\zeta_1 + \zeta_3) \ , \qquad \zeta'_4 = \tfrac{1}{\sqrt2}(\zeta_2\,,-\bar{\mathsf{z}}^2) \equiv (\zeta_2 + \zeta_4) \ .$$

Setting

$$\gamma_\lambda := \gamma(\tau_\lambda) \in \mathrm{End}(\boldsymbol{W})$$

one recovers the usual Weyl and Dirac representations as the matrices $\big(\gamma_\lambda\big)$, $\lambda = 0,1,2,3$, in the Weyl and Dirac bases respectively.

1.6 Further structures

Some other operations on 4-spinor space, commonly used in the literature, actually depend on particular choices or conventions. Similarly to the choice of a basis or of a gauge they are useful in certain arguments or calculations, but don't need to be fixed in the theory's foundations. I'll describe the cases of a Hermitian form on $\boldsymbol{U}$, of *charge conjugation*, *parity* and *time reversal*; I'll show the relations among these objects and how they are related to the notion of *observer*.

A Hermitian 2-form h on $\boldsymbol{U}$ is an element in $\overline{\boldsymbol{U}}^\star \vee \boldsymbol{U}^\star$, hence it can be seen as an element in $\boldsymbol{H}^*$; more precisely, $\bar h \in \boldsymbol{H}^*$. One says that h is *normalized* if it is non-degenerate, positive and $g^\#(h) = h^{-1}$; the latter condition is equivalent to $g(h,h) = 2$. If h is normalized then it is necessarily a future-pointing timelike element in $\boldsymbol{H}^*$. For example, consider the Pauli basis (τ_λ) determined by a normalized 2-spinor basis (ζ_A), and let (t^λ) be the dual basis; then $\sqrt2\,\bar{\mathsf{t}}^0 = \bar{\mathsf{z}}^1 \otimes \mathsf{z}^1 + \bar{\mathsf{z}}^2 \otimes \mathsf{z}^2$ is normalized; conversely, every positive-definite normalized Hermitian metric h can be expressed in the above form for some suitable normalized 2-spinor bases.[6]

[6]Similarly, negative-definite Hermitian metrics correspond to past-pointing timelike covectors. Hermitian metrics of mixed signature $(1,-1)$ correspond to spacelike covectors; actually, such metrics can always be written as proportional to $\sqrt2\,\bar{\mathsf{t}}^3 = \bar{\mathsf{z}}^1 \otimes \mathsf{z}^1 - \bar{\mathsf{z}}^2 \otimes \mathsf{z}^2$, in appropriate normalized 2-spinor bases.

The basic observation resulting from the above discussion is that the assignments of an 'observer' in $\boldsymbol{H}$ and of a positive-definite Hermitian metric on $\boldsymbol{U}$ are equivalent; actually, the two objects are nearly the same thing. In 4-spinor terms, the above equivalence is only slightly less obvious. If h is assigned, then it extends naturally to a Hermitian metric h on $\boldsymbol{W}$, which can be characterized by[7]

$$h(\psi,\phi)=\mathrm{k}(\gamma_0\psi,\phi)\ .$$

Charge conjugation depends on the choice of a normalized 2-form $\omega=\mathrm{e}^{\mathrm{i}t}\,\varepsilon\in\wedge^2\boldsymbol{U}^\star$, and is defined as the anti-isomorphism

$$\mathcal{C}_\omega:\boldsymbol{W}\to\boldsymbol{W}:\psi\mapsto\mathcal{C}_\omega(\psi)\equiv\mathcal{C}(u,\chi)=\big(\omega^\#(\bar\chi),-\bar\omega^\flat(\bar u)\big)=\mathrm{e}^{-\mathrm{i}t}\,\big(\varepsilon^\#(\bar\chi),-\bar\varepsilon^\flat(\bar u)\big)\ .$$

Thus $\mathcal{C}_\omega=\mathrm{e}^{-\mathrm{i}t}\,\mathcal{C}_\varepsilon$. One also gets

$$\begin{aligned}&\mathcal{C}_\omega\circ\mathcal{C}_\omega=1\!\!1_{\boldsymbol{W}}\ ,\\&\gamma_y\circ\mathcal{C}_\omega+\mathcal{C}_\omega\circ\gamma_y=0\quad\Leftrightarrow\quad\mathcal{C}_\omega\circ\gamma_y\circ\mathcal{C}_\omega=-\gamma_y\ ,\quad y\in\boldsymbol{H}\,.\end{aligned}$$

Finally, parity is an isomorphism of $\boldsymbol{W}$ dependent on the choice of an observer, while time-reversal is an anti-isomorphism dependent on the choice of an observer *and* of a normalized 2-form; they are defined by

$$\mathcal{P}:=\gamma_0\equiv\gamma(\tau_0)\ ,\qquad\mathcal{T}_\omega:=\gamma_\eta\gamma_0\mathcal{C}_\omega\ ,$$

where the chosen observer is expressed as τ_0 in a suitable Pauli basis, and γ_η is the canonical element of the Dirac algebra corresponding to the g-normalized volume form of $\boldsymbol{H}$, and expressed in a Pauli basis as $\gamma_\eta=\gamma_0\gamma_1\gamma_2\gamma_3$ (see §4.1).

Remark. An observer, seen as a Hermitian metric on $\boldsymbol{U}$, also determines an isomorphism $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\to\boldsymbol{U}\otimes\boldsymbol{U}^\star\equiv\mathrm{End}(\boldsymbol{U})$. Through it, one can view 'world spinors' as endomorphisms, thus recovering the algebraic structure for the Galileian treatment of spin [1].

1.7 2-spinor groups

The group $\mathrm{Aut}(\boldsymbol{S})\cong\mathrm{Aut}(\boldsymbol{U})\subset\boldsymbol{U}\otimes\boldsymbol{U}^\star$ has the natural subgroups

$$\begin{aligned}&\mathrm{Sl}(\boldsymbol{U}):=\{K\in\mathrm{Aut}(\boldsymbol{U}):\det K=1\}\ ,&&\dim_{\mathbb{C}}\mathrm{Sl}(\boldsymbol{U})=3\ ,\\&\mathrm{Sl}^c(\boldsymbol{U}):=\{K\in\mathrm{Aut}(\boldsymbol{U}):|\det K|=1\}\ ,&&\dim_{\mathbb{R}}\mathrm{Sl}^c(\boldsymbol{U})=7\ .\end{aligned}$$

The former is the group of all automorphisms of $\boldsymbol{S}$ (of $\boldsymbol{U}$) which leave any complex volume form invariant; the latter is the group of all automorphisms which leave any complex volume form invariant up to a phase factor, and thus it can be seen as the group which preserves the two-spinor structure. One has the Lie algebras

$$\begin{aligned}&\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\cong\{A\in\mathrm{End}(\boldsymbol{U}):\mathrm{Tr}\,A=0\}\ ,\\&\mathfrak{L}\mathrm{Sl}^c(\boldsymbol{U})\cong\{A\in\mathrm{End}(\boldsymbol{U}):\Re\,\mathrm{Tr}\,A=0\}=\mathrm{i}\,\mathbb{R}\oplus\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\ .\end{aligned}$$

[7] In the traditional notation, $\gamma_\lambda^\dagger$ indicates the h-adjoint of γ_λ, and then depends on the chosen observer.

If $h \in \boldsymbol{U}^\star \bar{\vee} \overline{\boldsymbol{U}}^\star$ is a positive Hermitian metric then one sets

$$\mathrm{U}(\boldsymbol{U}, h) := \{K \in \mathrm{Aut}(\boldsymbol{U}) : K^\dagger = K^{-1}\} \subset \mathrm{Sl}^c(\boldsymbol{U}) ,$$

$$\mathrm{SU}(\boldsymbol{U}, h) := \{K \in \mathrm{Aut}(\boldsymbol{U}) : K^\dagger = K^{-1} ,\ \det K = 1\} \subset \mathrm{Sl}(\boldsymbol{U}) ,$$

where $K^\dagger$ denotes the h-adjoint of K. One gets the Lie algebras

$$\mathfrak{L}\mathrm{U}(\boldsymbol{U}, h) = \{A \in \mathrm{End}(\boldsymbol{U}) : A + A^\dagger = 0\} = \mathrm{i}\,\mathbb{R} \oplus \mathfrak{L}\mathrm{SU}(\boldsymbol{U}, h) ,$$

$$\mathfrak{L}\mathrm{SU}(\boldsymbol{U}, h) = \{A \in \mathrm{End}(\boldsymbol{U}) : A + A^\dagger = 0 ,\ \ \mathrm{Tr}\, A = 0\} .$$

Now observe that $\mathrm{End}(\boldsymbol{U})$ can be decomposed into the direct sum of the subspaces of all h-Hermitian and anti-Hermitian endomorphisms; the restriction of this decomposition to $\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})$ gives then

$$\mathfrak{L}\mathrm{Sl}(\boldsymbol{U}) = \mathfrak{L}\mathrm{SU}(\boldsymbol{U}, h) \oplus \mathrm{i}\,\mathfrak{L}\mathrm{SU}(\boldsymbol{U}, h) .$$

When a 2-spinor basis is fixed, then one gets group isomorphisms $\mathrm{Sl}(\boldsymbol{U}) \to \mathrm{Sl}(2, \mathbb{C})$, $\mathrm{SU}(\boldsymbol{U}, h) \to \mathrm{SU}(2)$ and the like.

1.8 2-spinor groups and Lorentz group

Up to an obvious transposition we can make the identification

$$\mathrm{End}(\boldsymbol{U}) \otimes \mathrm{End}(\overline{\boldsymbol{U}}) \cong \mathrm{End}(\boldsymbol{U} \otimes \overline{\boldsymbol{U}}) .$$

We then write[8]

$$(K \otimes \bar{H})^{AA^{\boldsymbol{\cdot}}}{}_{BB^{\boldsymbol{\cdot}}} = K^A{}_B\, \bar{H}^{A^{\boldsymbol{\cdot}}}{}_{B^{\boldsymbol{\cdot}}} , \quad K \in \mathrm{End}(\boldsymbol{U}) ,$$

$$(K \otimes \bar{H})^\lambda{}_\mu = K^A{}_B\, \bar{H}^{A^{\boldsymbol{\cdot}}}{}_{B^{\boldsymbol{\cdot}}} \tau^\lambda{}_{AA^{\boldsymbol{\cdot}}} \tau_\mu{}^{BB^{\boldsymbol{\cdot}}} .$$

The group $\mathrm{Aut}(\boldsymbol{U}) \times \mathrm{Aut}(\overline{\boldsymbol{U}})$ can be identified with the subgroup of $\mathrm{Aut}(\boldsymbol{U} \otimes \overline{\boldsymbol{U}})$ constituted of all elements of the type $K \otimes \bar{H}$ with $K, H \in \mathrm{Aut}\,\boldsymbol{U}$. This subgroup is sometimes written as $\mathrm{Aut}(\boldsymbol{U}) \otimes \mathrm{Aut}(\overline{\boldsymbol{U}})$, which of course must not be intended as a true tensor product. It has the proper subgroup $\mathrm{Aut}(\boldsymbol{U}) \bar{\vee} \mathrm{Aut}(\overline{\boldsymbol{U}})$, constituted of all automorphisms of the type $K \otimes \bar{K}$, $K \in \mathrm{Aut}(\boldsymbol{U})$.

Proposition 1.4 $\mathrm{Aut}(\boldsymbol{U}) \bar{\vee} \mathrm{Aut}(\overline{\boldsymbol{U}})$ *preserves the splitting* $\boldsymbol{U} \otimes \overline{\boldsymbol{U}} = \boldsymbol{H} \oplus \mathrm{i}\,\boldsymbol{H}$ *and the causal structure of* $\boldsymbol{H}$.

PROOF: There exist bases of $\boldsymbol{H}$ composed of isotropic elements; these are also complex bases of isotropic elements of $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$. Then $A \in \mathrm{Aut}(\boldsymbol{U} \otimes \overline{\boldsymbol{U}})$ preserves the splitting and the causal structure iff it sends any element of the form $u \otimes \bar{u}$ in an element of the form $v \otimes \bar{v}$. □

[8] The elements of the dual Pauli basis can be written as $\mathsf{t}^\lambda = \tau^\lambda{}_{AA^{\boldsymbol{\cdot}}} \mathsf{z}^A \otimes \bar{\mathsf{z}}^{A^{\boldsymbol{\cdot}}}$ with $\tau^\lambda{}_{AA^{\boldsymbol{\cdot}}} = g^{\lambda\mu}\, \varepsilon^{AB}\, \bar{\varepsilon}^{A^{\boldsymbol{\cdot}}B^{\boldsymbol{\cdot}}} \tau_\mu{}^{BB^{\boldsymbol{\cdot}}}$.

Accordingly, on sets

$$\mathrm{Sl}^c(\boldsymbol{U})\,\bar{\vee}\,\mathrm{Sl}^c(\overline{\boldsymbol{U}})=\mathrm{Sl}(\boldsymbol{U})\,\bar{\vee}\,\mathrm{Sl}(\overline{\boldsymbol{U}}):=\{K\otimes\bar{K}:K\in\mathrm{Sl}(\boldsymbol{U})\}\ .$$

Since K preserves ε up to a phase factor, $K\otimes\bar{K}$ preserves $\varepsilon\otimes\bar{\varepsilon}\equiv g$; moreover it is immediate to check that any Pauli basis is transformed to another Pauli basis. From proposition 1.2 it then follows that $\mathrm{Sl}(\boldsymbol{U})\,\bar{\vee}\,\mathrm{Sl}(\overline{\boldsymbol{U}})$ restricted to $\boldsymbol{H}$ coincides with the special ortochronous Lorentz group $\mathrm{Lor}^{\uparrow}_{+}(\boldsymbol{H},g)$. Actually, the epimorphism $\mathrm{Sl}(\boldsymbol{U})\to\mathrm{Lor}^{\uparrow}_{+}(\boldsymbol{H},g)$ turns out to be 2-to-1.

The Lie algebra of $\mathrm{Sl}(\boldsymbol{U})\,\bar{\vee}\,\mathrm{Sl}(\overline{\boldsymbol{U}})$ is the Lie subalgebra of $\mathrm{End}(\boldsymbol{U})\otimes\mathrm{End}(\overline{\boldsymbol{U}})$ constituted by all elements which can be written in the form

$$A\otimes 1\!\!1_{\overline{\boldsymbol{U}}}+1\!\!1_{\boldsymbol{U}}\otimes\bar{A}\ ,\quad A\in\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\ .$$

One easily checks that these restrict to endomorphisms of $\boldsymbol{H}$, actually they constitute the vector space of all g-antisymmetric endomorphisms of $\boldsymbol{H}$ namely the Lie algebra $\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)$. Let a normalized 2-spinor basis be fixed; then the isomorphism $\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\leftrightarrow\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)$, taking into account the isomorphism $\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)\leftrightarrow\wedge^2\boldsymbol{H}^*$ induced by the Lorentz metric g, associates the basis $(\nu_i\,;\check{\nu}_i)$ with the basis $(\rho_i\,;\check{\rho}_i)$, $i=1,2,3$, where[9]

$$\nu_i:=-\mathrm{i}\,\check{\nu}_i \qquad \check{\nu}_i:=\tfrac{1}{2}\,\sigma_i\equiv\tfrac{1}{2}\,\sigma_i{}^A{}_B\,\zeta_A\otimes\mathsf{z}^B\ ,\ ,$$

$$\rho_i:=-{*}\check{\rho}_i\ , \qquad \check{\rho}_i:=2\,\mathsf{t}^0\wedge\mathsf{t}^i\ .$$

A Hermitian metric h on $\boldsymbol{U}$, besides the above said (§1.7) splitting of $\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})$, also determines an "observer" $\tau_0:=\frac{1}{\sqrt{2}}\,\bar{h}^{\#}$, hence also the splitting of $\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)$ into "infinitesimal rotations" and "infinitesimal boosts" as

$$\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)=\mathfrak{L}\mathrm{Lor}_{\mathrm{R}}(\boldsymbol{H},g,\tau_0)\oplus\mathfrak{L}\mathrm{Lor}_{\mathrm{B}}(\boldsymbol{H},g,\tau_0)\ .$$

If one chooses a normalized 2-spinor basis such that the element τ_0 of the corresponding Pauli basis of $\boldsymbol{H}$ coincides with the given observer, then the bases $(\nu_i\,;\check{\nu}_i)$ and $(\rho_i\,;\check{\rho}_i)$ turn out to be adapted to the respective splittings.

Remark. On $\mathfrak{L}\mathrm{Lor}(\boldsymbol{H},g)$ one has the pseudo-metric induced by g; moreover, consider the real symmetric 2-form

$$\mathrm{K}_{\mathfrak{L}\mathrm{Sl}}:\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\times\mathfrak{L}\mathrm{Sl}(\boldsymbol{U})\to\mathbb{R}:(A,B)\mapsto 2\,\Re\,\mathrm{Tr}(A\circ B)\ .$$

Then it turns out that the bases $(\nu_i\,;\check{\nu}_i)$ and $(\rho_i\,;\check{\rho}_i)$ are orthonormal, and that the signature of both metrics is $(-,-,-,+,+,+)$. So, the splittings of the two algebras determined by the choice of an "observer" can't be into arbitrary subspaces: the two components must be mutually orthogonal subspaces of opposite signature.

2 Two-spinor bundles

2.1 Two-spinor connections

Consider any real manifold $\boldsymbol{M}$ and a vector bundle $\boldsymbol{S}\to\boldsymbol{M}$ with complex 2-dimensional fibres. Denote base manifold coordinates as (x^a); choose a local frame

[9] Here again $(\sigma_i{}^A{}_B)$ denotes the i-th Pauli matrix. (t^λ) is the dual Pauli basis. Also note that the Hodge isomorphism restricts to a complex structure on $\wedge^2\boldsymbol{H}^*$.

(ξ_A) of $\boldsymbol{S}$, determining linear fibre coordinates (x^A). According to the constructions of the previous sections, one now has the bundles $\boldsymbol{Q}$, $\mathbb{L}$, $\boldsymbol{U}$, $\boldsymbol{H}$ over $\boldsymbol{M}$, with smooth natural structures; the frame (ξ_A) yields the frames ε, l, (ζ_A) and (τ_λ), respectively. Moreover for any rational number $r \in \mathbb{Q}$ one has the semi-vector bundle $\mathbb{L}^r$.

Consider an arbitrary $\mathbb{C}$-linear connection $\digamma$ on $\boldsymbol{S} \to \boldsymbol{M}$, called a 2-*spinor connection.* In the fibred coordinates $(\mathsf{x}^a, \mathsf{x}^A)$ $\digamma$ is expressed by the coefficients $\digamma_a{}^A{}_B : \boldsymbol{M} \to \mathbb{C}$, namely the covariant derivative of a section $s : \boldsymbol{M} \to \boldsymbol{S}$ is expressed as

$$\nabla s = (\partial_a s^A - \digamma_a{}^A{}_B s^B)\,\mathrm{d}\mathsf{x}^a \otimes \xi_A \ .$$

The rule $\nabla\bar{s} = \overline{\nabla s}$ yields a connection $\bar{\digamma}$ on $\overline{\boldsymbol{S}} \to \boldsymbol{M}$, whose coefficients are given by

$$\bar{\digamma}_a{}^{\dot{A}}{}_{\dot{B}} = \overline{\digamma_a{}^A{}_B} \ .$$

Actually, $\digamma$ determines linear connections on each of the above said induced vector bundles over $\boldsymbol{M}$ (in particular, it is easy to see that any $\mathbb{C}$-linear connection on a complex vector bundle determines a $\mathbb{R}$-linear connection on the induced Hermitian tensor bundle). Denote by $2\,G$ and $2\,Y$ the connections induced on $\mathbb{L}$ and $\boldsymbol{Q}$ (this notation makes sense because the fibres are 1-dimensional), namely

$$\nabla l = -2\,G_a\,\mathrm{d}\mathsf{x}^a \otimes l \ , \quad \nabla\varepsilon = 2\,\mathrm{i}\,Y_a\,\mathrm{d}\mathsf{x}^a \otimes \varepsilon \ ,$$
$$\nabla \mathsf{w}^{-1} \equiv \nabla(l^{-1} \otimes \varepsilon) = 2(G_a + \mathrm{i}\,Y_a)\,\mathrm{d}\mathsf{x}^a \otimes l^{-1} \otimes \varepsilon$$

and the like. By direct calculation we find

$$G_a = \Re(\tfrac{1}{2}\,\digamma_a{}^A{}_A) = \tfrac{1}{4}(\digamma_a{}^A{}_A + \bar{\digamma}_a{}^{\dot{A}}{}_{\dot{A}}) \ ,$$

$$Y_a = \Im(\tfrac{1}{2}\,\digamma_a{}^A{}_A) = \tfrac{1}{4\mathrm{i}}(\digamma_a{}^A{}_A - \bar{\digamma}_a{}^{\dot{A}}{}_{\dot{A}}) \ .$$

Note that since Y_a are real the induced linear connection on $\boldsymbol{Q}$ is Hermitian (preserves its natural Hermitian structure).

The coefficients of the connection $\tilde{\digamma}$ induced on $\boldsymbol{U}$ are given by

$$\tilde{\digamma}_a{}^A{}_B = \digamma_a{}^A{}_B - G_a\,\delta^A{}_B \ .$$

Let $\tilde{\Gamma}$ be the connection induced on $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$, and Γ' the connection induced on $\boldsymbol{S} \otimes \overline{\boldsymbol{S}}$. Then

$$\Gamma'{}_a{}^{A\dot{A}}{}_{B\dot{B}} = \digamma_a{}^A{}_B\,\delta^{\dot{A}}{}_{\dot{B}} + \delta^A{}_B\,\bar{\digamma}_a{}^{\dot{A}}{}_{\dot{B}} \ ,$$

$$\tilde{\Gamma}_a{}^{A\dot{A}}{}_{B\dot{B}} = \digamma_a{}^A{}_B\,\delta^{\dot{A}}{}_{\dot{B}} + \delta^A{}_B\,\bar{\digamma}_a{}^{\dot{A}}{}_{\dot{B}} - 2\,G_a\,\delta^A{}_B\,\delta^{\dot{A}}{}_{\dot{B}} \ .$$

Since the above coefficients are real, Γ' and $\tilde{\Gamma}$ turn out to be reducible to real connections on $\boldsymbol{S} \mathbin{\bar{\vee}} \overline{\boldsymbol{S}}$ and $\boldsymbol{H} \equiv \boldsymbol{U} \mathbin{\bar{\vee}} \overline{\boldsymbol{U}}$, respectively. Moreover

Proposition 2.1 *The connection $\tilde{\Gamma}$ induced on $\boldsymbol{H}$ by any 2-spinor connection is metric, namely $\nabla[\tilde{\Gamma}]g = 0$.*

PROOF: The Lorentz metric g of $\boldsymbol{H}$ can be identified with the identity of the bundle $\mathbb{L}^{-2}$, namely it is the canonical section $1 \equiv \varepsilon^{-1} \otimes \varepsilon : \boldsymbol{M} \to \mathbb{L}^{-2} \otimes \mathbb{L}^2 \equiv \boldsymbol{M} \times \mathbb{R}^+$, which obviously has vanishing covariant derivative. □

Because of metricity the coefficients $\tilde{\Gamma}_a{}^\lambda{}_\mu$ of $\tilde{\Gamma}$ in the frame (τ_λ) are antisymmetric and traceless, namely

$$\tilde{\Gamma}_a{}^{\lambda\mu} + \tilde{\Gamma}_a{}^{\mu\lambda} = 0 \;, \quad \tilde{\Gamma}_a{}^\lambda{}_\lambda = 0$$

(the second formula says $\nabla\eta = 0$, where η is the g-normalized volume form of $\boldsymbol{H}$).

The above relations between Ɣ and the induced connections can be inverted as follows:

Proposition 2.2 *One has*

$$Ɣ_a{}^A{}_B = (-G_a + \mathrm{i}\,Y_a)\,\delta^A{}_B + \tfrac{1}{2}\,\Gamma'_a{}^{A\dot{A}}{}_{B\dot{A}} = (G_a + \mathrm{i}\,Y_a)\,\delta^A{}_B + \tfrac{1}{2}\,\tilde{\Gamma}_a{}^{A\dot{A}}{}_{B\dot{A}} \;.$$

In 4-spinor formalism the above relation reads

$$Ɣ_a{}^\alpha{}_\beta = (G_a + \mathrm{i}\,Y_a)\,\delta^\alpha{}_\beta + \tfrac{1}{4}\,\tilde{\Gamma}_a{}^{\lambda\mu}(\gamma_\lambda\,\gamma_\mu)^\alpha{}_\beta \;,$$

where now $Ɣ_a{}^\alpha{}_\beta$ stands for the coefficients of the naturally induced connection $(Ɣ, \bar{Ɣ}^\star)$ on $\boldsymbol{W} \equiv \boldsymbol{U} \oplus_{\boldsymbol{M}} \overline{\boldsymbol{U}}^\star$ in any 4-spinor frame, $\alpha, \beta = 1, .., 4$.

A similar relation holds among the curvature tensors, namely

$$R_{ab}{}^A{}_B = 2\,(\mathrm{d}G - \mathrm{i}\,\mathrm{d}Y)_{ab}\,\delta^A{}_B + \tfrac{1}{2}\,R'_{ab}{}^{A\dot{A}}{}_{B\dot{A}} =$$
$$= -2\,(\mathrm{d}G + \mathrm{i}\,\mathrm{d}Y)_{ab}\,\delta^A{}_B + \tfrac{1}{2}\,\tilde{R}_{ab}{}^{A\dot{A}}{}_{B\dot{A}} \;,$$

where R, R' and $\tilde{R}$ are the curvature tensors of Ɣ, Γ' and $\tilde{\Gamma}$, respectively.

Remark. Under a local gauge transformation $\mathsf{K} : \boldsymbol{M} \to \mathrm{Gl}(2,\mathbb{C})$ the above coefficients transform as

$$Ɣ_a{}^A{}_B \mapsto (\mathsf{K}^{-1})^A_C\,\mathsf{K}^D_B\,Ɣ_a{}^C{}_D - (\mathsf{K}^{-1})^A_C\,\partial_a\mathsf{K}^C_B \;,$$
$$G_a \mapsto G_a - \tfrac{1}{2}\,\partial_a \log|\det \mathsf{K}| \;, \quad Y_a \mapsto Y_a - \tfrac{1}{2}\,\partial_a \arg\det\mathsf{K} \;,$$
$$\tilde{\Gamma}_a{}^\lambda{}_\mu \mapsto (\tilde{\mathsf{K}}^{-1})^\lambda_\nu\,\tilde{\mathsf{K}}^\rho_\mu\,\tilde{\Gamma}_a{}^\nu{}_\rho - (\tilde{\mathsf{K}}^{-1})^\lambda_\nu\,\partial_a\tilde{\mathsf{K}}^\nu_\mu \;.$$

2.2 Two-spinor tetrad

Henceforth I'll assume that $\boldsymbol{M}$ is a real 4-dimensional manifold. Consider a linear morphism

$$\Theta : \mathrm{T}\boldsymbol{M} \to \boldsymbol{S} \otimes \overline{\boldsymbol{S}} = \mathbb{C} \otimes \mathbb{L} \otimes \boldsymbol{H} \;,$$

namely a section

$$\Theta : \boldsymbol{M} \to \mathbb{C} \otimes \mathbb{L} \otimes \boldsymbol{H} \otimes \mathrm{T}^*\boldsymbol{M}$$

(all tensor products are over $\boldsymbol{M}$). Its coordinate expression is

$$\Theta = \Theta^\lambda_a\,\tau_\lambda \otimes \mathrm{d}\mathsf{x}^a = \Theta^{A\dot{A}}_a\,\zeta_A \otimes \bar{\zeta}_{\dot{A}} \otimes \mathrm{d}\mathsf{x}^a \;, \qquad \Theta^\lambda_a, \Theta^{A\dot{A}}_a : \boldsymbol{M} \to \mathbb{C} \otimes \mathbb{L} \;.$$

We'll assume that Θ is non-degenerate and valued in the Hermitian subspace $\mathbb{L} \otimes \boldsymbol{H} \subset \boldsymbol{S} \otimes \overline{\boldsymbol{S}}$; then Θ can be viewed as a 'scaled' *tetrad* (or *soldering form*, or *vierbein*); the coefficients Θ^λ_a are real (i.e. valued in $\mathbb{R} \otimes \mathbb{L}$) while the coefficients $\Theta^{A\dot{A}}_a$ are Hermitian, i.e. $\bar{\Theta}^{\dot{A}A}_a = \Theta^{A\dot{A}}_a$.

Remark. Most of what follows actually still holds in the case of a degenerate tetrad. The inverse Θ^{-1} is not used. This will give rise to a more natural theory, in which all field equations are of the first order. Possible degeneracy might also have a physical meaning, as discussed in [4].

Through a tetrad, the geometric structure of the fibres of $\boldsymbol{H}$ is carried to a similar, scaled structure on the fibres of $\mathrm{T}\boldsymbol{M}$. It will then be convenient, from now on, to distinguish by a tilda the objects defined on $\boldsymbol{H}$, so I'll denote by $\tilde{g}$, $\tilde{\eta}$ and $\tilde{\gamma}$ the Lorentz metric, the $\tilde{g}$-normalized volume form and the Dirac map of $\boldsymbol{H}$, and set

$$g := \Theta^*\tilde{g} : \boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^2\otimes\mathrm{T}^*\boldsymbol{M}\otimes\mathrm{T}^*\boldsymbol{M} \ ,$$

$$\eta := \Theta^*\tilde{\eta} : \boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^4\otimes\wedge^4\mathrm{T}^*\boldsymbol{M} \ ,$$

$$\gamma := \tilde{\gamma}\circ\Theta : \mathrm{T}\boldsymbol{M} \to \mathbb{L}\otimes\mathrm{End}(\boldsymbol{W}) \ ,$$

which have the coordinate expressions

$$g = \eta_{\lambda\mu}\,\Theta^\lambda_a\,\Theta^\mu_b\,\mathrm{dx}^a\otimes\mathrm{dx}^b = \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\,\Theta^{A\dot{A}}_a\,\Theta^{B\dot{B}}_b\,\mathrm{dx}^a\otimes\mathrm{dx}^b \ ,$$

$$\eta = \det(\Theta)\,\mathrm{dx}^0\wedge\mathrm{dx}^1\wedge\mathrm{dx}^2\wedge\mathrm{dx}^3 \ ,$$

$$\gamma = \sqrt{2}\,\Theta^{A\dot{A}}_a\,(\zeta_A\otimes\bar{\zeta}_{\dot{A}} + \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}\,\bar{\mathsf{z}}^{\dot{B}}\otimes\mathsf{z}^B)\otimes\mathrm{dx}^a \ .$$

The above objects turn out to be a Lorentz metric, the corresponding volume form and a Clifford map. Moreover

$$\Theta^b_\mu := \Theta^\lambda_a\,\eta_{\lambda\mu}\,g^{ab} = (\Theta^{-1})^b_\mu : \boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^{-1} \ , \quad g^{ab} : \boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^{-2} \ .$$

A non-degenerate tetrad, together with a two-spinor frame, yields mutually dual orthonormal frames (Θ_λ) of $\mathbb{L}^{-1}\otimes\mathrm{T}\boldsymbol{M}$ and $(\overset{*}{\Theta}{}^\lambda)$ of $\mathbb{L}\otimes\mathrm{T}^*\boldsymbol{M}$, given by

$$\Theta_\lambda := \Theta^{-1}(\tau_\lambda) = \Theta^a_\lambda\,\partial\mathsf{x}_a \ , \quad \overset{*}{\Theta}{}^\lambda := \Theta^*(\mathsf{t}^\lambda) = \Theta^\lambda_a\,\mathrm{dx}^a \ .$$

We also write

$$\gamma = \gamma_\lambda\otimes\overset{*}{\Theta}{}^\lambda = \gamma_a\otimes\mathrm{dx}^a \ , \quad \gamma_\lambda := \gamma(\Theta_\lambda) : \boldsymbol{M} \to \mathrm{End}(\boldsymbol{W}) \ ,$$
$$\gamma_a := \gamma(\partial\mathsf{x}_a) = \Theta^\lambda_a\,\gamma_\lambda : \boldsymbol{M} \to \mathbb{L}\otimes\mathrm{End}(\boldsymbol{W}) \ .$$

2.3 Cotetrad

One defines a natural 'exterior' product of elements in the fibres of $\boldsymbol{H}\otimes_{\boldsymbol{M}}\mathrm{T}^*\boldsymbol{M}$ by requiring that, for decomposable tensors, it is given by

$$(y_1\otimes\alpha_1)\wedge(y_2\otimes\alpha_2) = (y_1\wedge y_2)\otimes(\alpha_1\wedge\alpha_2) \ , \quad \alpha_1\,,\alpha_2\in\mathrm{T}^*\boldsymbol{M}\,,\ u_1\,,u_2\in\boldsymbol{H} \ .$$

We'll consider the exterior products

$$\wedge^q\Theta : \boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^q\otimes\wedge^q\boldsymbol{H}\otimes\wedge^q\mathrm{T}^*\boldsymbol{M} \ , \quad q = 1,2,3,4 \ .$$

In particular, one has $\wedge^2\Theta \equiv \Theta\wedge\Theta$, that is

$$\wedge^2\Theta(u\wedge v) = \Theta(u)\wedge\Theta(v) \quad \Rightarrow \quad \wedge^2\Theta = \Theta^\lambda_a\Theta^\mu_b\,(\tau_\lambda\wedge\tau_\mu)\otimes(\mathrm{dx}^a\wedge\mathrm{dx}^b) \ .$$

Next, consider the linear map over $\boldsymbol{M}$

$$\breve{\Theta} : (\boldsymbol{S}\otimes\overline{\boldsymbol{S}})\otimes \mathrm{T}^*\boldsymbol{M} \to \mathbb{C}\otimes\mathbb{L}^4\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}$$

defined by

$$\breve{\Theta}(\xi) := \tfrac{1}{3!}\,\tilde{\eta}\mid(\xi\wedge\Theta\wedge\Theta\wedge\Theta) = \tfrac{1}{3!}\,\tilde{\eta}\mid[\xi\wedge(\wedge^3\Theta)]\ .$$

Its coordinate expression is

$$\breve{\Theta}(\xi) = \breve{\Theta}^a_\lambda\,\xi^\lambda_a\,\mathrm{d}^4\mathsf{x} := \tfrac{1}{3!}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\Theta^\mu_b\Theta^\nu_c\Theta^\rho_d\,\xi^\lambda_a\,\mathrm{d}^4\mathsf{x}\ ,$$

$$\xi = \xi^\lambda_a\,\tau_\lambda\otimes\mathrm{d}\mathsf{x}^a\ ,\quad \xi^\lambda_a : \boldsymbol{M}\to\mathbb{C}\otimes\mathbb{L}\ .$$

Now $\breve{\Theta}$ can be seen as a bilinear map $(\boldsymbol{S}\otimes\overline{\boldsymbol{S}})\times\mathrm{T}^*\boldsymbol{M}\to\mathbb{C}\otimes\mathbb{L}^4\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}$ over $\boldsymbol{M}$, or also as a linear map

$$\boldsymbol{S}\otimes\overline{\boldsymbol{S}} \to \mathbb{C}\otimes\mathbb{L}^4\otimes\mathrm{T}\boldsymbol{M}\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}$$

over $\boldsymbol{M}$. Using the latter point of view, if Θ is non-degenerate then one has

$$\breve{\Theta} = \Theta^{-1}\otimes\eta\ .$$

Namely, in general one may regard $\breve{\Theta}$, which is called the *co-tetrad*, as a kind of 'pseudo-inverse' of Θ, defined even if Θ is degenerate.

The above construction can be easily generalized, for $p = 0,1,2,3,4$, to a map

$$\breve{\Theta}^{(p)} : \wedge^p(\boldsymbol{S}\otimes\overline{\boldsymbol{S}})\otimes(\wedge^p\mathrm{T}^*\boldsymbol{M}) \to \mathbb{C}\otimes\mathbb{L}^4\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}\ .$$

We'll be concerned with $\breve{\Theta}^{(1)} = \breve{\Theta}$ and $\breve{\Theta}^{(2)}$. Note that $\breve{\Theta}^{(0)} = \eta$.

2.4 Tetrad and connections

If Ɽ is a complex-linear connection on $\boldsymbol{S}$, and G and $\tilde{\Gamma}$ are the induced connections on $\mathbb{L}$ and $\boldsymbol{H}$, then a non-degenerate tetrad $\Theta : \mathrm{T}\boldsymbol{M}\to\mathbb{L}\otimes\boldsymbol{H}$ yields a unique connection Γ on $\mathrm{T}\boldsymbol{M}$, characterized by the condition

$$\nabla[\Gamma\otimes\tilde{\Gamma}]\Theta = 0\ .$$

Moreover Γ is metric, i.e. $\nabla[\Gamma]g = 0$. Denoting by $\Gamma_a{}^\lambda{}_\mu$ the coefficients of Γ in the frame $\Theta'_\lambda \equiv \Theta^{-1}(l\otimes\tau_\lambda)$ one obtains

$$\Gamma_a{}^\lambda{}_\mu = \tilde{\Gamma}_a{}^\lambda{}_\mu + 2\,G_a\,\delta^\lambda{}_\mu\ .$$

The curvature tensors of Γ and $\tilde{\Gamma}$ are related by $R_{ab}{}^\lambda{}_\mu = \tilde{R}_{ab}{}^\lambda{}_\mu$, or

$$R_{ab}{}^c{}_d = \tilde{R}_{ab}{}^\lambda{}_\mu\,\Theta^c_\lambda\,\Theta^\mu_d\ .$$

Hence the Ricci tensor and the scalar curvature are given by

$$R_{ad} = R_{ab}{}^b{}_d = \tilde{R}_{ab}{}^\lambda{}_\mu\,\Theta^b_\lambda\,\Theta^\mu_d\ ,$$
$$R_a{}^a = \tilde{R}_{ab}{}^{\lambda\mu}\,\Theta^b_\lambda\,\Theta^a_\mu\ .$$

In general, the connection Γ will have non-vanishing torsion,[10] which can be expressed[11] as

$$\Theta^\lambda_c\,T^c{}_{ab} = \partial_{[a}\Theta^\lambda_{b]} + \Theta^\mu_{[a}\,\tilde\Gamma_{b]}{}^\lambda{}_\mu + 2\,\Theta^\lambda_{[a}\,G_{b]}\ .$$

Remark. The torsion can be seen as the Frölicher-Nijenhuis bracket

$$\tilde T := T\rfloor\Theta = [\Gamma',\Theta] : \boldsymbol{M} \to \wedge^2\mathrm{T}^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} H'\ ,$$

where $H' = \mathbb{L}\otimes\boldsymbol{H}$, $\Gamma' : \boldsymbol{H'} \to \mathrm{T}^*\boldsymbol{M}\otimes_{\boldsymbol{H'}}\mathrm{T}\boldsymbol{H'}$ is the induced connection on $\boldsymbol{H'}\to\boldsymbol{M}$, and Θ is seen as a vertical-valued form $\Theta : \boldsymbol{H'} \to \mathrm{T}^*\boldsymbol{M}\otimes_{\boldsymbol{H'}}\mathrm{V}\boldsymbol{H'}$.

2.5 The Dirac operator

Given a tetrad and a two-spinor connection, one introduces the Dirac operator acting on sections $\psi : \boldsymbol{M} \to \mathbb{L}^{-3/2}\otimes\boldsymbol{W}$.

Writing $\tilde\gamma^\# : \boldsymbol{M}\to\boldsymbol{H}\otimes\mathrm{End}(\boldsymbol{W})$, $\nabla\psi : \boldsymbol{M}\to\mathbb{L}^{-3/2}\otimes\mathrm{T}^*\boldsymbol{M}\otimes_{\boldsymbol{M}}\boldsymbol{W}$, one has

$$\tilde\gamma^\#\nabla\psi : \boldsymbol{M}\to\mathbb{L}^{-3/2}\otimes\boldsymbol{H}\otimes\mathrm{T}^*\boldsymbol{M}\otimes\boldsymbol{W}\ ,$$

where contraction in $\boldsymbol{W}$ is understood. Next, one contracts the factors $\boldsymbol{H}$ and $\mathrm{T}^*\boldsymbol{M}$ above via

$$\breve\Theta : \boldsymbol{M}\to\mathbb{C}\otimes\mathbb{L}^3\otimes\boldsymbol{H}^*\otimes\mathrm{T}\boldsymbol{M}\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}\ ,$$

obtaining

$$\breve{\nabla\!\!\!/}\psi := \langle\breve\Theta,\tilde\gamma^\#\nabla\psi\rangle : \boldsymbol{M}\to\mathbb{L}^{3/2}\otimes\boldsymbol{W}\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}\ ,$$

which has the coordinate expression

$$\breve{\nabla\!\!\!/}\psi = \breve\Theta^a_\lambda\,\Big(\sigma^{\lambda AA^{\cdot}}\nabla_a\chi_{A^{\cdot}}\zeta_A\,,\ \sigma^\lambda{}_{AA^{\cdot}}\nabla_a u^A\,\bar{\mathsf{z}}^{A^{\cdot}}\Big)\otimes\mathrm{d}^4\mathsf{x}\ .$$

This definition works even if Θ were degenerate; in the non-degenerate case one simply has $\breve{\nabla\!\!\!/}\psi = \nabla\!\!\!/\psi\otimes\eta$.

3 Two-spinors and field theories

3.1 The fields

In this section I'll present a "minimal geometric data" field theory: actually, the unique "geometric datum" is a vector bundle $\boldsymbol{S}\to\boldsymbol{M}$ with complex 2-dimensional fibres and real 4-dimensional base manifold. All other bundles and fixed geometric objects are determined just by this datum through functorial constructions, as we saw in the previous sections; no further background structure is assumed. Any considered bundle section which is not functorially fixed by our geometric datum is a field. In this way one obtains a field theory which turns out to be essentially equivalent to a classical theory of Einstein-Cartan-Maxwell-Dirac fields.

The fields are taken to be the tetrad Θ, the 2-spinor connection Ƒ, the electromagnetic field F and the electron field ψ. The gravitational field is represented by

[10] This is the tensor field $T : \boldsymbol{M}\to\mathrm{T}\boldsymbol{M}\otimes\wedge^2\mathrm{T}^*\boldsymbol{M}$ defined by $T(u,v) = \nabla_u v - \nabla_v u - [u,v]$, where $u,v : \boldsymbol{M}\to\mathrm{T}\boldsymbol{M}$ are any two vector fields, and has the coordinate expression $T^c{}_{ab} = -\Gamma_a{}^c{}_b + \Gamma_b{}^c{}_a$.

[11] Taking into account $0 = \nabla_a\Theta^\lambda_b = \partial_a\Theta^\lambda_b - \Gamma_a{}^\lambda{}_\mu\,\Theta^\mu_b + \Gamma_a{}^c{}_b\,\Theta^\lambda_c$.

Θ (which can be viewed as a 'square root' of the metric) and the traceless part of F, namely $\tilde{\Gamma}$, seen as the gravitational part of the connection. If Θ is non-degenerate one obtains, as in the standard metric-affine approach [10, 11, 13, 8], essentially the Einstein equation and the equation for torsion; the metricity of the spacetime connection is a further consequence. But note that the theory is non-singular also in the degenerate case. The connection G induced on $\mathbb{L}$ will be assumed to have vanishing curvature, $\mathrm{d}G = 0$, so that one can always find local charts such that $G_a = 0$; this amounts to gauging away the conformal ('dilaton') symmetry. Coupling constants will arise as covariantly constant sections of $\mathbb{L}$, which now becomes just a vector space.

The Dirac field is a section

$$\psi : \boldsymbol{M} \to \mathbb{L}^{-3/2} \otimes \boldsymbol{W} := \mathbb{L}^{-3/2} \otimes (\boldsymbol{U} \oplus \overline{\boldsymbol{U}}^{\star}) ,$$

assumed to represent a semiclassical particle with one-half spin, mass $m \in \mathbb{L}^{-1}$ and charge $q \in \mathbb{R}$.

The electromagnetic potential can be thought of as the Hermitian connection Y on $\wedge^2\boldsymbol{U}$ determined by F, whose coefficients are indicated as $\mathrm{i}\,Y_a$; locally one writes

$$Y_a \equiv q\,A_a ,$$

where $A : \boldsymbol{M} \to \mathrm{T}^*\boldsymbol{M}$ is a local 1-form.

The electromagnetic field is represented by a spinor field

$$\tilde{F} : \boldsymbol{M} \to \mathbb{L}^{-2} \otimes \wedge^2\boldsymbol{H}^*$$

which, via Θ, determines the 2-form $F := \Theta^*\tilde{F} : \boldsymbol{M} \to \wedge^2\mathrm{T}^*\boldsymbol{M}$. The relation between Y and F will follow as one of the field equations; note how this setting allows a first-order linear Lagrangian and non-singularity in the degenerate case also for the electromagnetic sector.

The total Lagrangian and the Euler-Lagrange operator will be the sum of a gravitational, an electromagnetic and a Dirac term

$$\mathcal{L} = \mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{em}} + \mathcal{L}_{\mathrm{D}} , \quad \mathcal{E} = \mathcal{E}_{\mathrm{g}} + \mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}} .$$

Observe that all Lagrangian 4-forms are defined in terms of the cotetrad $\breve{\Theta}$, while a direct translation of the standard formulation in terms of our fields would force one to use Θ^{-1}, resulting in a less simple and natural theory.

3.2 Gravitational Lagrangian

The tetrad Θ and the curvature tensor $\tilde{R}$ of $\tilde{\Gamma}$ can be assembled into a 4-form $\mathcal{L}_{\mathrm{g}}$ which, in the non-degenerate case, turns out to be the usual gravitational Lagrangian density:

$$\mathcal{L}_{\mathrm{g}} := \frac{1}{4\,\Bbbk}\,\breve{\Theta}^{(2)}(\tilde{R}^{\#}) = \frac{1}{8\,\Bbbk}\,\tilde{\eta} \mid (\tilde{R}^{\#} \wedge \Theta \wedge \Theta) : \boldsymbol{M} \to \wedge^4\mathrm{T}^*\boldsymbol{M} ,$$

where $\tilde{R}^{\#} : \boldsymbol{M} \to \wedge^2\mathrm{T}^*\boldsymbol{M} \otimes \wedge^2\boldsymbol{H}$ is the curvature tensor of $\tilde{\Gamma}$ with one index raised via $\tilde{g}$, and $\Bbbk \in \mathbb{L}^2$ is Newton's gravitational constant. Note how this is necessary in

order to obtain a true (non-scaled) 4-form on $\boldsymbol{M}$ and the correct coupling with the spinor field. One has the coordinate expression $\mathcal{L}_{\mathrm{g}} = \ell_{\mathrm{g}}\,\mathrm{d}^4\mathsf{x}$ with

$$\ell_{\mathrm{g}} = \frac{1}{8\,\Bbbk}\,\varepsilon_{\lambda\mu\nu\rho}\,\varepsilon^{abcd}\,\tilde{R}_{ab}{}^{\lambda\mu}\,\Theta^{\nu}_{c}\,\Theta^{\rho}_{d} = \frac{1}{2\,\Bbbk}\,R\,\det\Theta\ ,$$

where R is the scalar curvature and the last equality holds if Θ is non-degenerate.

A calculation gives the Θ- and $\tilde{\Gamma}$-components of the gravitational part $\mathcal{E}_{\mathrm{g}}$ of the Euler-Lagrange operator:

$$(\mathcal{E}_{\mathrm{g}})^{c}_{\nu} = \frac{1}{4\,\Bbbk}\,\varepsilon_{\lambda\mu\nu\rho}\,\varepsilon^{abcd}\,R_{ab}{}^{\lambda\mu}\,\theta^{\rho}_{d}\ ,$$

$$(\mathcal{E}_{\mathrm{g}})^{a}{}_{\lambda\mu} = \frac{1}{2\,\Bbbk}\,\varepsilon_{\lambda\mu\nu\rho}\,\varepsilon^{abcd}\,(\partial_b\Theta^{\nu}_{c} + \Theta^{\sigma}_{b}\,\tilde{\Gamma}_{c\,\sigma}{}^{\nu}\,)\,\Theta^{\rho}_{d}\ .$$

In the non-degenerate case these are essentially the Einstein tensor and the torsion of the spacetime connection, respectively. The first, in particular, can be written

$$(\mathcal{E}_{\mathrm{g}})^{c}_{\nu} = \frac{1}{4\,\Bbbk}\,\Theta^{a}_{[\lambda}\,\Theta^{b}_{\mu}\,\Theta^{c}_{\nu]}\,\det\Theta = \frac{1}{\Bbbk}\,(R_{ab}{}^{bc} - \tfrac{1}{2}\,R_{db}{}^{bd}\,\delta^{c}_{a})\Theta^{a}_{\nu}\,\det\Theta\ .$$

The $\tilde{\Gamma}$-component of $\mathcal{E}_{\mathrm{g}}$ can be expressed in terms of the torsion as

$$(\mathcal{E}_{\mathrm{g}})^{a}{}_{\lambda\mu} = \frac{1}{4\,\Bbbk}\,\varepsilon_{\lambda\mu\nu\rho}\,\varepsilon^{abcd}\,T^{e}{}_{bc}\,\Theta^{\nu}_{e}\,\Theta^{\rho}_{d}\ .$$

3.3 Electromagnetic Lagrangian

The electromagnetic potential and the Maxwell field will be considered independent fields. The former is represented by a local section $A : \boldsymbol{M} \to \mathrm{T}^*\boldsymbol{M}$, related to the connection Y induced by F on $\wedge^2\boldsymbol{U}$ by the relation $Y = q\,A$. The Maxwell field is a section $\tilde{F} : \boldsymbol{M} \to \mathbb{L}^{-2} \otimes \wedge^2\boldsymbol{H}^*$, written in coordinates as $\tilde{F} = \tilde{F}_{\lambda\mu}\,\mathsf{t}^{\lambda} \otimes \mathsf{t}^{\mu}$. The e.m. Lagrangian density is defined to be

$$\mathcal{L}_{\mathrm{em}} = \ell_{\mathrm{em}}\,\mathrm{d}^4\mathsf{x} = \Big[-\tfrac{1}{2}\,\Theta^{(2)}(\mathrm{d}A \otimes \tilde{F}) + \tfrac{1}{4}\,(\tilde{F}\cdot\tilde{F})\Big]\,\eta\ ,$$

with coordinate expression

$$\ell_{\mathrm{em}} = -\tfrac{1}{4}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\partial_a A_b\,\tilde{F}^{\lambda\mu}\,\Theta^{\nu}_{c}\Theta^{\rho}_{d} + \tfrac{1}{4}\,\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}\,\det\Theta\ .$$

In the non-degenerate case, this turns out to be essentially the Lagrangian used in the ADM formalism.

Since $\tilde{F}$ does not appear in the other terms of the total Lagrangian, the $\tilde{F}$-component of the field equations is immediately seen to yield

$$-\tfrac{1}{2}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\partial_a A_b\,\Theta^{\nu}_{c}\,\Theta^{\rho}_{d} + \tilde{F}_{\lambda\mu}\,\det\Theta = 0\ ,$$

which in the non-degenerate case gives

$$F := \Theta^*\tilde{F} = 2\,\mathrm{d}A \quad\Rightarrow\quad \mathcal{L}_{\mathrm{em}} = -\tfrac{1}{4}\,F^2\,\eta\ .$$

The A-component of the Euler-Lagrange operator is

$$\begin{aligned}(\mathcal{E}_{\mathrm{em}})^{a} &= \tfrac{1}{2}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,(\partial_b\tilde{F}^{\lambda\mu}\,\Theta^{\nu}_{c}\,\Theta^{\rho}_{d} + 2\,\tilde{F}^{\lambda\mu}\,\partial_b\Theta^{\nu}_{c}\,\Theta^{\rho}_{d}) =\\ &= \tfrac{1}{2}\,\varepsilon^{abcd}\,(\mathrm{d}{*}F)_{bcd}\ .\end{aligned}$$

The Θ-component is

$$(\mathcal{E}_{\rm em})^c_\nu = -\tfrac{1}{2}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\partial_a A_b\,\tilde{F}^{\lambda\mu}\Theta^\rho_d + \tfrac{1}{4}\,\tilde{F}^2\,\breve{\Theta}^c_\nu\ ,$$

which in the non-degenerate case becomes essentially the usual Maxwell stress-energy tensor

$$(\mathcal{E}_{\rm em})^c_\nu = (F_{ab}F^{ac} - \tfrac{1}{4}\,F^2\,\delta^c_b)\breve{\Theta}^b_\nu\ .$$

3.4 Dirac Lagrangian

The Dirac spinor field and its 'Dirac adjoint' are sections

$$\begin{aligned}\psi &= (u,\chi) : \boldsymbol{M} \to \mathbb{L}^{-3/2}\otimes\boldsymbol{W} = \mathbb{L}^{-3/2}\otimes(\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^\star)\ ,\\ \bar\psi &= (\bar\chi,\bar u) : \boldsymbol{M} \to \mathbb{L}^{-3/2}\otimes(\boldsymbol{U}^\star\oplus\overline{\boldsymbol{U}}) = \mathbb{L}^{-3/2}\otimes\boldsymbol{W}^\star\ .\end{aligned}$$

In coordinates:

$$\begin{aligned}&u = u^A\,\zeta_A\ ,\quad \chi = \chi_{A^\cdot}\bar{\mathsf{z}}^{A^\cdot}\ ,\quad u^A\,,\chi_{A^\cdot} : \boldsymbol{M}\to\mathbb{C}\otimes\mathbb{L}^{-3/2}\\ &\langle\bar\psi,\psi\rangle = (\bar u^{A^\cdot}\chi_{A^\cdot} + \bar\chi_A\,u^A) : \boldsymbol{M}\to\mathbb{C}\otimes\mathbb{L}^{-3}\ .\end{aligned}$$

The Dirac operator (§2.5) yields a section

$$\breve{\nabla\!\!\!/}\psi : \boldsymbol{M}\to\mathbb{L}^{3/2}\otimes\boldsymbol{W}\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}\ ,$$

so that

$$\langle\bar\psi,\breve{\nabla\!\!\!/}\psi\rangle : \boldsymbol{M}\to\mathbb{C}\otimes\wedge^4\mathrm{T}^*\boldsymbol{M}\ .$$

Now we introduce the scalar density

$$\mathcal{L}_{\rm D} = \tfrac{\mathrm{i}}{2}\left(\langle\bar\psi,\breve{\nabla\!\!\!/}\psi\rangle - \langle\breve{\nabla\!\!\!/}\bar\psi,\psi\rangle\right) - m\,\langle\bar\psi\,,\psi\rangle\,\eta : \boldsymbol{M}\to\wedge^4\mathrm{T}^*\boldsymbol{M}\ ,$$

where $\breve{\nabla\!\!\!/}\bar\psi := \overline{\breve{\nabla\!\!\!/}\psi}$, and $m\in\mathbb{L}^{-1}$ is the described particle's mass. This is a version of the Dirac Lagrangian which remains non-singular when Θ is degenerate. In the non-degenerate case one also has

$$\mathcal{L}_{\rm D} = \left[\tfrac{\mathrm{i}}{2}\,\left(\langle\bar\psi,\nabla\!\!\!/\psi\rangle - \langle\nabla\!\!\!/\bar\psi,\psi\rangle\right) - m\,\langle\bar\psi\,,\psi\rangle\right]\eta\ ;$$

in 2-spinor terms this reads

$$\mathcal{L}_{\rm D} = \tfrac{\mathrm{i}}{\sqrt2}\breve{\Theta}\Big(\nabla u\otimes\bar u - u\otimes\nabla\bar u + \tilde g^\#(\bar\chi\otimes\nabla\chi - \nabla\bar\chi\otimes\chi)\Big) - m\,\Big(\langle\chi,\bar u\rangle + \langle\bar\chi,u\rangle\Big)\,\eta\ ,$$

with the coordinate expression

$$\begin{aligned}\ell_{\rm D} = \tfrac{\mathrm{i}}{\sqrt2}\,\breve{\Theta}^a_{AA^\cdot}\Big(\nabla_a u^A\,\bar u^{A^\cdot} - u^A\,\nabla_a\bar u^{A^\cdot} + \varepsilon^{AB}\bar\varepsilon^{A^\cdot B^\cdot}(\bar\chi_B\,\nabla_a\chi_{B^\cdot} - \nabla_a\bar\chi_B\,\chi_{B^\cdot})\Big)&\\ - m\,(\bar\chi_A u^A + \chi_{A^\cdot}\bar u^{A^\cdot})\,\det\Theta\ .&\end{aligned}$$

Next we compute the Euler-Lagrange operator $\mathcal{E}_{\rm D}$. The $\bar u$-component is

$$(\mathcal{E}_{\rm D})_{A^\cdot} = \sqrt2\,\mathrm{i}\,\breve{\Theta}^a_{AA^\cdot}\nabla_a u^A - m\,\chi_{A^\cdot}\,\det\Theta + \tfrac{\mathrm{i}}{\sqrt2}\,T_{AA^\cdot}u^A\ ,$$

where $T_{AA^\bullet} := \breve{\Theta}^a_{AA^\bullet} T^b{}_{ab}$ is used for replacing the term with $\partial_a \Theta^\mu_b$ (see §2.4).

The $\bar{\chi}$-component is

$$(\mathcal{E}_{\mathrm{D}})^A = \sqrt{2}\,\mathrm{i}\,\breve{\Theta}^{aAA^\bullet}\nabla_a \chi_{A^\bullet} - m\,u^A\,\det\Theta + \tfrac{\mathrm{i}}{\sqrt{2}}\,T^{AA^\bullet}\chi_{A^\bullet}\,,$$

with $\breve{\Theta}^{aAA^\bullet} := \breve{\Theta}^a_{BB^\bullet}\varepsilon^{BA}\bar{\varepsilon}^{B^\bullet A^\bullet}$ and $T^{AA^\bullet} := \varepsilon^{BA}\bar{\varepsilon}^{B^\bullet A^\bullet} T_{BB^\bullet}$.

The $\tilde{\Gamma}$-component is

$$\begin{aligned}(\mathcal{E}_{\mathrm{D}})^a_{\lambda\mu} = \tfrac{\mathrm{i}}{4\sqrt{2}}\,[(\breve{\Theta}^a_{AC^\bullet}\,\tau_{[\lambda}{}^{DC^\bullet}\tau_{\mu]DA^\bullet} - \breve{\Theta}^a_{CA^\bullet}\,\tau_{[\lambda}{}^{CD^\bullet}\tau_{\mu]AD^\bullet})u^A\bar{u}^{A^\bullet} \\ + (\breve{\Theta}^{aBC^\bullet}\tau_{[\lambda}{}^{DB^\bullet}\tau_{\mu]DC^\bullet} - \breve{\Theta}^{aCB^\bullet}\tau_{[\lambda}{}^{BD^\bullet}\tau_{\mu]CD^\bullet})\bar{\chi}_B\chi_{B^\bullet}]\,.\end{aligned}$$

The Θ-component is

$$\begin{aligned}(\mathcal{E}_{\mathrm{D}})^c_\nu = \varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\Theta^\mu_b\,\Theta^\rho_d\Big[\\ \tfrac{\mathrm{i}}{2\sqrt{2}}\left(\nabla_a u^A\,\bar{u}^{A^\bullet} - u^A\,\nabla_a\bar{u}^{A^\bullet} + \varepsilon^{BA}\bar{\varepsilon}^{B^\bullet A^\bullet}(\bar{\chi}_B\nabla_a\chi_{B^\bullet} - \nabla_a\bar{\chi}_B\chi_{B^\bullet})\right)\tau^\lambda{}_{AA^\bullet} \\ - \tfrac{1}{3!}\,m\,(\bar{\chi}_A\,u^A + \chi_{A^\bullet}\bar{u}^{A^\bullet})\,\Theta^\lambda_a\Big] = \\ = \tfrac{\mathrm{i}}{4}\,\varepsilon^{abcd}\,\varepsilon_{\lambda\mu\nu\rho}\,\Theta^\mu_b\,\Theta^\rho_d\Big(\bar{\psi}\tilde{\gamma}^\lambda\nabla_a\psi - \bar{\tilde{\gamma}}^\lambda\nabla_a\bar{\psi}\,\psi\Big) - m\,\bar{\psi}\psi\,\breve{\Theta}^c_\nu\,.\end{aligned}$$

The A-component is simply

$$(\mathcal{E}_{\mathrm{D}})^a = \sqrt{2}\,q\,\breve{\Theta}^a_{AA^\bullet}\Big(u^A\,\bar{u}^{A^\bullet} + \varepsilon^{BA}\,\bar{\varepsilon}^{B^\bullet A^\bullet}\,\bar{\chi}_B\,\chi_{B^\bullet}\Big) = q\,\breve{\Theta}^a_\lambda\,(\bar{\psi}\tilde{\gamma}^\lambda\psi)\,.$$

3.5 Field equations

Having calculated the various pieces of $\mathcal{E} = \mathcal{E}_{\mathrm{g}} + \mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}}$, writing down the field equations $\mathcal{E} = 0$ is a simple matter. These equations are non-singular also when Θ is degenerate; in the non-degenerate case one expects this approach to reproduce essentially the usual Einstein-Cartan-Maxwell-Dirac field equations.

The Θ-component

$$(\mathcal{E}_{\mathrm{g}})^c_\nu = -(\mathcal{E}_{\mathrm{em}} + \mathcal{E}_{\mathrm{D}})^c_\nu\,,$$

corresponds to the Einstein equation; actually, as already discussed, in the non-degenerate case the left-hand side is essentially the Einstein tensor, while the right-hand side can be viewed as the sum of the energy-momentum tensors of the electromagnetic field and of the Dirac field.

The $\tilde{\Gamma}$-component gives the equation for torsion

$$(\mathcal{E}_{\mathrm{g}})^a{}_{\lambda\mu} = -(\mathcal{E}_{\mathrm{D}})^a{}_{\lambda\mu}\,.$$

From this one sees that the spinor field is a source for torsion, and that in this context one cannot formulate a torsion-free theory.

It was already seen (§3.3) that the $\tilde{F}$-component reads $F = 2\,\mathrm{d}A$ in the non-degenerate case, and of course this yields the first Maxwell equation $\mathrm{d}F = 0$. The A-component is

$$-\tfrac{1}{2}\,\varepsilon^{abcd}(\mathrm{d}{*}F)_{bcd} + q\,\breve{\Theta}^a_\lambda\,(\bar{\psi}\tilde{\gamma}^\lambda\psi) = 0 \quad \text{i.e.} \quad \tfrac{1}{2}\,c\,\varepsilon^{abcd}(\mathrm{d}{*}F)_{bcd} = q\,\breve{\Theta}^a_\lambda\,(\bar{\psi}\tilde{\gamma}^\lambda\psi)\,.$$

In the non-degenerate case this gives the second Maxwell equation

$$\tfrac{1}{2}\,{*}\mathrm{d}{*}F = j\ ,$$

where $j : \boldsymbol{M} \to\ \otimes\mathrm{T}^*\boldsymbol{M}$ is the *Dirac current*, with coordinate expression

$$j := \frac{q}{c}\,\Theta_a^\lambda\,(\bar\psi\tilde\gamma_\lambda\psi)\,\mathrm{dx}^a\ .$$

The $\bar u$- and $\bar\chi$-components $(\mathcal{E}_{\mathrm{D}})_A = 0$ and $(\mathcal{E}_{\mathrm{D}})^{\dot B} = 0$ give the following generalized form of the standard *Dirac equation*:

$$\begin{cases}\sqrt{2}\,\mathrm{i}\,\breve\Theta^a_{A\dot A}\nabla_a u^A - m\,\chi_{\dot A}\,\det\Theta + \frac{\mathrm{i}}{\sqrt{2}}\,T_{A\dot A}\,u^A = 0\\ \sqrt{2}\,\mathrm{i}\,\breve\Theta^{aA\dot A}\nabla_a\chi_{\dot A} - m\,u^A\,\det\Theta + \frac{\mathrm{i}}{\sqrt{2}}\,T^{A\dot A}\,\chi_{\dot A} = 0\end{cases}.$$

Denoting by $\breve T$ the 1-form obtained from the torsion by contraction, with coordinate expression $\breve T_a \equiv T_a = T^b{}_{ab}$, the above equation can be written in coordinate-free form as

$$\Big(\mathrm{i}\,\nabla\!\!\!/ - m + \tfrac{\mathrm{i}}{2}\,\gamma^\#(\breve T)\Big)\,\psi = 0\ .$$

4 Dirac algebra in two-spinor terms

4.1 Dirac algebra

If $\boldsymbol{V}$ is a finite-dimensional real vector space endowed with a non-degenerate scalar product, then its *Clifford algebra* $\boldsymbol{C}(\boldsymbol{V})$ is the associative algebra generated by $\boldsymbol{V}$ where the product of any $u, v \in \boldsymbol{V}$ is subjected to the condition

$$u\,v + v\,u = 2\,u\cdot v\ ,\quad u, v \in \boldsymbol{V}\ .$$

The Clifford algebra fulfills the following *universal property*: if $\boldsymbol{A}$ is an associative algebra with unity and $\gamma : \boldsymbol{V} \to \boldsymbol{A}$ is a linear map such that $\gamma(v)\,\gamma(v) = v{\cdot}v\ \forall v \in \boldsymbol{V}$, then γ extends to a unique homomorphism $\hat\gamma : \boldsymbol{C}(\boldsymbol{V}) \to \boldsymbol{A}$. It turns out that $\boldsymbol{C}(\boldsymbol{V})$ is isomorphic, as a vector space, to the vector space underlying the exterior algebra $\wedge\boldsymbol{V}$; through this isomorphism one identifies $v_1\wedge\ldots\wedge v_p$ with the antisymmetrized Clifford product

$$\frac{1}{p!}\,\big(v_1 v_2{\cdot\cdot}v_p - v_2 v_1{\cdot\cdot}v_p + \cdots\ \big)$$

where the sum is extended to all permutations of the set $\{1,\ldots,p\}$, with the appropriate signs. In other terms, one has two distinct algebras on the same underlying vector space: any element of $\boldsymbol{C}(\boldsymbol{V})$ can be uniquely expressed as a sum of terms, each of well-defined exterior degree. For example, one has $u\,v = u\wedge v + u\cdot v$; from this one sees that the Clifford algebra product does not preserve the exterior algebra degree, but only its parity: $\boldsymbol{C}(\boldsymbol{V})$ is $\mathbb{Z}_2$-graded. If $\phi \in \wedge^r\boldsymbol{V}$, $\theta \in \wedge^s\boldsymbol{V}$, then the Clifford product $\phi\,\theta$ turns out to be a sum of terms of exterior degree $r{+}s,\ r{+}s{-}2\,,\ldots,|r{-}s|$.

The Clifford algebra $\boldsymbol{D} := \boldsymbol{C}(\boldsymbol{H})$ of Minkowski space $\boldsymbol{H}$ (§1.4) is called the *Dirac algebra*. The Dirac map $\gamma : \boldsymbol{H} \to \mathrm{End}(\boldsymbol{W})$ is a Clifford map, hence by virtue of the above said universal property one can see the Dirac algebra as a real vector

subspace $\boldsymbol{D}\subset\operatorname{End}(\boldsymbol{W})$ of dimension $2^4=16$. Since this coincides with the *complex* dimension of $\operatorname{End}(\boldsymbol{W})\equiv\boldsymbol{W}\otimes\boldsymbol{W}^\star$, one gets $\operatorname{End}(\boldsymbol{W})=\mathbb{C}\otimes\boldsymbol{D}$.

The Dirac algebra $\boldsymbol{D}$ is multiplicatively generated by $\gamma(\boldsymbol{H})\subset\operatorname{End}(\boldsymbol{W})$, simply identified with $\boldsymbol{H}$. One has the natural decompositions

$$\boldsymbol{D}=\boldsymbol{D}^{(+)}\oplus\boldsymbol{D}^{(-)}=\big(\mathbb{R}\oplus\wedge^2\boldsymbol{H}\oplus\wedge^4\boldsymbol{H}\big)\oplus\big(\boldsymbol{H}\oplus\wedge^3\boldsymbol{H}\big)\ ,$$

where $\boldsymbol{D}^{(+)}$ and $\boldsymbol{D}^{(-)}$ denote the even-degree and odd-degree subspaces, respectively (the former is a subalgebra). Also, one has the distinguished elements

$$1\equiv1\!\!1_{\boldsymbol{W}}\subset\mathbb{R}\subset\boldsymbol{D}^{(+)}\ ,\quad \eta^\#\subset\wedge^4\boldsymbol{H}\subset\boldsymbol{D}^{(+)}\ ,$$

where $\eta^\#\equiv g^\#(\eta)$ is the contravariant tensor corresponding to the unimodular volume form η. One gets

$$\eta^\#\,\eta^\#=-1\ ,\quad \vartheta\,\eta^\#=*\vartheta\ \ \forall\vartheta\in\wedge\boldsymbol{H}\ ,$$

where $*$ is the Hodge isomorphism.

4.2 Decomposition of End $\boldsymbol{W}$ and ε-transposition

One has the natural decomposition

$$\operatorname{End}(\boldsymbol{W})\equiv\operatorname{End}(\boldsymbol{U}\oplus\overline{\boldsymbol{U}}{}^\star)=(\boldsymbol{U}\otimes\boldsymbol{U}^\star)\oplus(\boldsymbol{U}\otimes\overline{\boldsymbol{U}})\oplus(\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star)\oplus(\overline{\boldsymbol{U}}{}^\star\otimes\overline{\boldsymbol{U}})\ .$$

Accordingly, any $\Phi\in\operatorname{End}(\boldsymbol{W})$ is a 4-uple of tensors, which will be conveniently written in matricial form as

$$\Phi=\begin{pmatrix}K & P\\ Q & J\end{pmatrix}\ ,\qquad K\in\boldsymbol{U}\otimes\boldsymbol{U}^\star\ ,\ P\in\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\ ,\ Q\in\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star\ ,\ J\in\overline{\boldsymbol{U}}{}^\star\otimes\overline{\boldsymbol{U}}\ .$$

We now introduce an operation which acts on each of the above 4 types of tensors in a similar way. This operation, called *ε-transposition*, is actually independent of the particular normalized $\varepsilon\in\wedge^2\boldsymbol{U}^\star$ chosen; it is defined by

$$\begin{aligned}
&\boldsymbol{U}\otimes\boldsymbol{U}^\star\to\boldsymbol{U}^\star\otimes\boldsymbol{U}:K\mapsto\tilde K:=\langle\varepsilon^\flat\otimes\varepsilon^\#,K\rangle=\varepsilon_{CA}\,K^C{}_D\,\varepsilon^{DB}\,\mathsf{z}^A\otimes\zeta_B\ ,\\
&\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\to\boldsymbol{U}^\star\otimes\overline{\boldsymbol{U}}{}^\star:P\mapsto\tilde P:=\langle\varepsilon^\flat\otimes\bar\varepsilon^\flat,P\rangle=\varepsilon_{CA}\,P^{C\dot D}\,\bar\varepsilon_{\dot D\dot B}\,\mathsf{z}^A\otimes\bar{\mathsf{z}}^{\dot B}\ ,\\
&\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star\to\overline{\boldsymbol{U}}\otimes\boldsymbol{U}:Q\mapsto\tilde Q:=\langle\bar\varepsilon^\#\otimes\varepsilon^\#,Q\rangle=\bar\varepsilon^{\dot C\dot A}\,Q_{\dot CD}\,\varepsilon^{DB}\,\zeta_A\otimes\bar\zeta_{\dot B}\ ,\\
&\overline{\boldsymbol{U}}{}^\star\otimes\overline{\boldsymbol{U}}\to\overline{\boldsymbol{U}}\otimes\overline{\boldsymbol{U}}{}^\star:J\mapsto\tilde J:=\langle\bar\varepsilon^\#\otimes\bar\varepsilon^\flat,J\rangle=\bar\varepsilon^{\dot C\dot A}\,J_{\dot C}{}^{\dot D}\,\bar\varepsilon_{\dot D\dot B}\,\bar\zeta_{\dot A}\otimes\bar{\mathsf{z}}^{\dot B}\ .
\end{aligned}$$

Namely, *ε-transposition* changes the position (either high or low) of both indices of the tensor it acts on. For elements in $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ or $\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star$ it essentially amounts to index lowering (resp. raising) by the Lorentz metric g in complexified Minkowski space; for invertible elements in $\boldsymbol{U}\otimes\boldsymbol{U}^\star\equiv\operatorname{End}(\boldsymbol{U})$ or $\overline{\boldsymbol{U}}{}^\star\otimes\overline{\boldsymbol{U}}\equiv\operatorname{End}(\overline{\boldsymbol{U}}{}^\star)$, ε-transposition amounts to

$$\tilde X=(\det X)\,(X^{-1})^\star\ ,$$

where the superscript $\star$ denotes standard transposition.

It is clear that ε-transposition can be similarly defined[12] on $\boldsymbol{U}^\star\otimes\boldsymbol{U}$, $\boldsymbol{U}^\star\otimes\overline{\boldsymbol{U}}{}^\star$, $\overline{\boldsymbol{U}}\otimes\boldsymbol{U}$ and $\overline{\boldsymbol{U}}\otimes\overline{\boldsymbol{U}}{}^\star$, and in all cases one gets

$$\tilde{\tilde{X}}=X\ ,\quad (\tilde{X})^\star=(X^\star)^\sim\ ,\quad \tilde{X}\,X^\star=X^\star\tilde{X}=(\det X)\,1\!\!1\ ,\quad \det X=\det\tilde{X}\ .$$

Remark. The determinant is uniquely defined, via any ε, also for elements in $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$, $\boldsymbol{U}^\star\otimes\overline{\boldsymbol{U}}{}^\star$, $\overline{\boldsymbol{U}}\otimes\boldsymbol{U}$ and $\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star$. In these cases, the determinant of a tensor equals one-half its Lorentz pseudo-norm.

Moreover, whenever the composition of tensors X and Y is defined, one has

$$(X\,Y)^\sim=\tilde{X}\,\tilde{Y}\ ,\quad \mathrm{Tr}(\tilde{X}\,\tilde{Y})=\mathrm{Tr}(X\,Y)\ .$$

Whenever A and B are tensors of the same type, one has

$$\det(A+B)=\det(A)+\det(B)+\mathrm{Tr}(A^\star\tilde{B})\ ,$$

where the *scalar product* $(A,B)\mapsto\mathrm{Tr}(A^\star\tilde{B})$ is *symmetric.*[13]

Proposition 4.1 *Let* $\Phi=\begin{pmatrix}K & P\\ Q & J\end{pmatrix}\in\boldsymbol{W}\otimes\boldsymbol{W}^\star$ *be non-singular. Then*

$$\det\Phi=(\det K)\,(\det J)+(\det P)\,(\det Q)-\mathrm{Tr}(K^\star\,\tilde{P}\,J^\star\,\tilde{Q})\ ,$$

$$(\det\Phi)\,\Phi^{-1}=\begin{pmatrix}(\det J)\,\tilde{K}^\star-\tilde{Q}^\star\,J\,\tilde{P}^\star & (\det P)\,\tilde{Q}^\star-\tilde{K}^\star\,P\,\tilde{J}^\star\\ (\det Q)\,\tilde{P}^\star-\tilde{J}^\star\,Q\,\tilde{K}^\star & (\det K)\,\tilde{J}^\star-\tilde{P}^\star\,K\,\tilde{Q}^\star\end{pmatrix}\ .$$

PROOF: It can be checked by a direct calculation, taking into account the above identities. $\square$

4.3 ε-adjoint and characterization of $\boldsymbol{D}$

If X is a tensor of any of the above types, then its ε-*adjoint* is the tensor

$$X^\ddagger:=\bar{\tilde{X}}\ .$$

Using this operation one defines the real involution

$$\ddagger:\boldsymbol{W}\otimes\boldsymbol{W}^\star\to\boldsymbol{W}\otimes\boldsymbol{W}^\star:\begin{pmatrix}K & P\\ Q & J\end{pmatrix}\mapsto\begin{pmatrix}J^\ddagger & Q^\ddagger\\ P^\ddagger & K^\ddagger\end{pmatrix}\ .$$

Proposition 4.2 $\boldsymbol{D}$ *and* $\mathrm{i}\,\boldsymbol{D}$ *are the eigenspaces of* $\ddagger$ *corresponding to eigenvalues* $+1$ *and* -1*, respectively. Namely,* $\boldsymbol{D}$ *is the real subspace of* $\boldsymbol{W}\otimes\boldsymbol{W}^\star$ *constituted by all endomorphisms which can be written in the form*

$$\begin{pmatrix}K & P\\ P^\ddagger & K^\ddagger\end{pmatrix}\ ,\qquad K\in\boldsymbol{U}\otimes\boldsymbol{U}^\star\ ,\quad P\in\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\ .$$

Moreover one has the following characterisations

$$\boldsymbol{D}^0\equiv\mathbb{R}=\left\{r\begin{pmatrix}1\!\!1_{\boldsymbol{U}} & 0\\ 0 & 1\!\!1_{\overline{\boldsymbol{U}}{}^\star}\end{pmatrix}\ ,\quad r\in\mathbb{R}\right\}\ ,$$

[12] One could introduce ε-transposition on further spaces such as $\boldsymbol{U}\otimes\boldsymbol{U}$, $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}{}^\star$ and so on. These extensions however would depend from the chosen normalized ε; phase factors cancel out only in the considered cases.

[13] On $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{U}}\otimes\boldsymbol{U}$ (resp. $\boldsymbol{U}^\star\otimes\overline{\boldsymbol{U}}{}^\star$ and $\overline{\boldsymbol{U}}{}^\star\otimes\boldsymbol{U}^\star$) this coincides with $2\,g$ (resp. $2\,g^\#$).

$$\boldsymbol{D}^1 \equiv \boldsymbol{H} = \left\{ \begin{pmatrix} 0 & P \\ P^\ddagger & 0 \end{pmatrix} \;, \quad P \in \boldsymbol{H} \right\} ,$$

$$\boldsymbol{D}^2 \equiv \wedge^2 \boldsymbol{H} = \left\{ \begin{pmatrix} K & 0 \\ 0 & K^\ddagger \end{pmatrix} \;, \quad K \in \boldsymbol{U} \otimes \boldsymbol{U}^\star \,, \; \operatorname{Tr} K = 0 \right\} ,$$

$$\boldsymbol{D}^3 \equiv \wedge^3 \boldsymbol{H} = \left\{ \begin{pmatrix} 0 & P \\ P^\ddagger & 0 \end{pmatrix} \;, \quad P \in \mathrm{i}\, \boldsymbol{H} \right\} ,$$

$$\boldsymbol{D}^4 \equiv \wedge^4 \boldsymbol{H} = \left\{ \mathrm{i}\, r \begin{pmatrix} 1\!\!1_{\boldsymbol{U}} & 0 \\ 0 & -1\!\!1_{\overline{\boldsymbol{U}}^\star} \end{pmatrix} \;, \quad r \in \mathbb{R} \right\} ,$$

$$\boldsymbol{D}^{(+)} = \boldsymbol{D}^0 \oplus \boldsymbol{D}^2 \oplus \boldsymbol{D}^4 = \left\{ \begin{pmatrix} K & 0 \\ 0 & K^\ddagger \end{pmatrix} \;, \quad K \in \boldsymbol{U} \otimes \boldsymbol{U}^\star \right\} ,$$

$$\boldsymbol{D}^{(-)} = \boldsymbol{D}^1 \oplus \boldsymbol{D}^3 = \left\{ \begin{pmatrix} 0 & P \\ P^\ddagger & 0 \end{pmatrix} \;, \quad P \in \boldsymbol{U} \otimes \overline{\boldsymbol{U}} \right\} .$$

PROOF: The Dirac map $\gamma : \boldsymbol{H} \to \operatorname{End} \boldsymbol{W}$ can be written as

$$\gamma : v \mapsto \begin{pmatrix} 0 & \sqrt{2}\, v \\ \sqrt{2}\, v^\ddagger & 0 \end{pmatrix} ,$$

whence the characterization of $\boldsymbol{D}^1$. It immediately follows that $\boldsymbol{D}^{(+)}$ is constituted by diagonal-block elements, while $\boldsymbol{D}^{(-)}$ is constituted by off-diagonal-block elements. The other characterizations can be checked by matrix calculations. □

5 Clifford group and its subgroups

5.1 Clifford group

Let $\boldsymbol{D}^\bullet := \boldsymbol{D} \cap \operatorname{Aut} \boldsymbol{W}$ be the group of all invertible elements in $\boldsymbol{D}$. The Clifford group $\mathrm{Cl} \equiv \mathrm{Cl}(\boldsymbol{W})$ is defined to be [7, 9] the subgroup of $\boldsymbol{D}^\bullet$ under whose adjoint action $\boldsymbol{H}$ is stable. In other terms, $\Phi \in \boldsymbol{D}^\bullet$ is an element of Cl iff

$$\operatorname{Ad}[\Phi] v \equiv \Phi\, \gamma(v)\, \Phi^{-1} \in \gamma(\boldsymbol{H}) \,, \quad \forall\, v \in \boldsymbol{H} \;.$$

Using proposition 4.1 we write the adjoint action as

$$(\det \Phi)\; \operatorname{Ad}[\Phi] v = \begin{pmatrix} K & P \\ P^\ddagger & K^\ddagger \end{pmatrix} \begin{pmatrix} 0 & V \\ V^\ddagger & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Y^\ddagger & X^\ddagger \end{pmatrix} =$$

$$= \begin{pmatrix} P\, V^\ddagger\, X + K\, V\, Y^\ddagger & P\, V^\ddagger\, Y + K\, V\, X^\ddagger \\ K^\ddagger\, V^\ddagger\, X + P^\ddagger\, V\, Y^\ddagger & K^\ddagger\, V^\ddagger\, Y + P^\ddagger\, V\, X^\ddagger \end{pmatrix} ,$$

where $V \equiv \sqrt{2}\, v$ and

$$X \equiv (\det \bar{K})\, \tilde{K}^\star - \bar{P}^\star\, \tilde{\bar{K}}\, \tilde{P}^\star \,, \qquad Y \equiv (\det P)\, \bar{P}^\star - \tilde{K}^\star\, P\, \bar{K}^\star \,,$$

$$X^\ddagger = (\det K)\, \bar{K}^\star - \tilde{P}^\star\, K\, \bar{P}^\star \,, \qquad Y^\ddagger = (\det \bar{P})\, \tilde{P}^\star - \bar{K}^\star\, \tilde{\bar{P}}\, \tilde{K}^\star \,.$$

Lemma 5.1 *An element of* $\boldsymbol{D}^\bullet$ *which belongs to the Clifford group is necessarily either odd or even, so that the Clifford group is the disjoint union* $\mathrm{Cl} = \mathrm{Cl}^{(+)} \cup \mathrm{Cl}^{(-)}$ *where* $\mathrm{Cl}^{(+)} \equiv \mathrm{Cl} \cap \boldsymbol{D}^{(+)}$, $\mathrm{Cl}^{(-)} \equiv \mathrm{Cl} \cap \boldsymbol{D}^{(-)}$.

PROOF: If Φ is in Cl then the $\boldsymbol{U} \otimes \boldsymbol{U}^\star$-component of $\mathrm{Ad}[\Phi]v$ vanishes for all $v \in \boldsymbol{H}$, namely

$$K\,V\,\tilde{\bar{Y}} = -P\,\tilde{\bar{V}}\,X\ , \quad \forall\, V \in \boldsymbol{H}\ .$$

Composing both sides with $\tilde{V}^\star\,\tilde{K}^\star$ on the left and with $\tilde{X}^\star$ on the right one finds

$$(\det K)\,(\det V)\,\tilde{\bar{Y}}\,\tilde{X}^\star = -(\det\Phi)(\det\bar{K})\,\tilde{V}^\star\,\tilde{K}^\star\,P\,\tilde{\bar{V}}\ .$$

Now the above equality is certainly fulfilled in the particular case when $\det K = 0$. Suppose $\det K \neq 0$ for the moment (the other case will be considered later). The left-hand side vanishes for all null elements $V \in \boldsymbol{H}$, thus also $\tilde{V}^\star\,\tilde{K}^\star\,P\,\tilde{\bar{V}}$ vanishes for all null vectors V; it's not difficult to see that this implies $\tilde{K}^\star\,P = 0$, which on turn implies $P = 0$. Summarizing, if $\Phi \in \mathrm{Cl}$ and $\det K \neq 0$ then $P = 0$. By a similar argument, composing the equation $K\,V\,\tilde{\bar{Y}} = -P\,\tilde{\bar{V}}\,X$ on the left by $\bar{V}^\star\,\tilde{P}^\star$ and on the right by $\bar{Y}^\star$, one finds that if $\Phi \in \mathrm{Cl}$ and $\det P \neq 0$ then $K = 0$.

The case which remains to be considered is that when $\det K = \det P = 0$. Since $\det P = \frac{1}{2}\,g(P,P)$, P is an isotropic element of $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$, and as such it is decomposable. Similarly, K is decomposable. Namely one can write

$$K = k \otimes \lambda\ , \quad P = p \otimes \bar{q}\ , \quad V = s \otimes \bar{s}\ , \qquad k,p,q,s \in \boldsymbol{U},\ \lambda \in \boldsymbol{U}^\star\ .$$

A little two-spinor algebra then yields

$$P\,\tilde{\bar{V}}\,X + K\,V\,\tilde{\bar{Y}} = \bar{\varepsilon}(\bar{k},\bar{p})\,\Big[\,\langle\lambda,q\rangle\,|\langle\lambda,s\rangle|^2\,k\otimes k^\flat - \langle\bar{\lambda},\bar{q}\rangle\,|\varepsilon(s,q)|^2\,p\otimes p^\flat\,\Big]\ ,$$

$$\det\Phi = -\,\mathrm{Tr}(K\,\bar{P}^\star\,\tilde{\bar{K}}\,\tilde{P}^\star\,) = |\varepsilon(k,p)|^2\,|\langle\lambda,q\rangle|^2\ .$$

Now one sees that in order that $\det\Phi \neq 0$ one must have $\langle\lambda,q\rangle \neq 0$ and $\varepsilon(k,p) \neq 0$. Thus $k\otimes k^\flat$ and $p\otimes p^\flat$ are linearly independent elements of $\boldsymbol{U} \otimes \boldsymbol{U}^\star$ and, in order that $P\,\tilde{\bar{V}}\,X + K\,V\,\tilde{\bar{Y}}$ vanishes for all V, one must have $\langle\lambda,s\rangle = \varepsilon(q,s)$ for all $s \in \boldsymbol{U}$, which implies $\lambda = 0$ and $q = 0$ that is $K = 0$ and $P = 0$, a contradiction. Thus the case $\det K = \det P = 0$ cannot yield an element $\Phi \in \mathrm{Cl}$. □

Proposition 5.1
a) $\mathrm{Cl}^{(+)}$ *is the* 7*-dimensional real submanifold of* $\boldsymbol{D}^{(+)}$ *constituted of all elements in* $\boldsymbol{W} \otimes \boldsymbol{W}^\star$ *which are of the type*

$$\begin{pmatrix} K & 0 \\ 0 & K^\ddagger \end{pmatrix}\ , \quad K \in \boldsymbol{U} \otimes \boldsymbol{U}^\star,\ \det K \in \mathbb{R} \setminus \{0\}\ .$$

b) $\mathrm{Cl}^{(-)}$ *is the* 7*-dimensional real submanifold of* $\boldsymbol{D}^{(-)}$ *constituted of all elements in* $\boldsymbol{W} \otimes \boldsymbol{W}^\star$ *which are of the type*

$$\begin{pmatrix} 0 & P \\ P^\ddagger & 0 \end{pmatrix}\ , \quad P \in \boldsymbol{U} \otimes \overline{\boldsymbol{U}},\ \det P \in \mathbb{R} \setminus \{0\}\ .$$

PROOF:
a) Let $\Phi = \left(\begin{smallmatrix} K & 0 \\ 0 & K^\ddagger \end{smallmatrix}\right)$, $K \in \boldsymbol{U} \otimes \boldsymbol{U}^\star$, $\det K \neq 0$. Then

$$(\det \Phi)\ \mathrm{Ad}[\Phi]v = \begin{pmatrix} 0 & (\det K)\, K\, V\, \bar{K}^\star \\ (\det \bar{K}) \tilde{\bar{K}}\, \tilde{\bar{V}}\ \ \tilde{K}^\star & 0 \end{pmatrix} \ , \quad V \equiv \sqrt{2}\, v \in \boldsymbol{H} \ .$$

For $\mathrm{Ad}[\Phi]v$ to be in $\boldsymbol{H}$, the two non-zero entries of the above matrix must be in $\boldsymbol{H} \equiv \boldsymbol{U} \,\bar{\vee}\, \overline{\boldsymbol{U}}$ and in $\overline{\boldsymbol{U}}^\star \,\bar{\vee}\, \boldsymbol{U}^\star$, respectively. Consider the $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$-entry. Since $\bar{V} = V^\star$ because V is Hermitian, one finds

$$[(\det K)\, K\, V\, \bar{K}^\star]^{\overline{\star}} = (\det \bar{K})\, K\, V\, \bar{K}^\star \ ,$$

and $(\det K)\, K\, V\, \bar{K}^\star$ is Hermitian for all $V \in \boldsymbol{H}$ iff $\det K = \det \bar{K}$ (this argument gives the same result for the other non-zero entry).
b) Let $\Phi = \left(\begin{smallmatrix} 0 & P \\ P^\ddagger & 0 \end{smallmatrix}\right)$, $P \in \boldsymbol{U} \otimes \overline{\boldsymbol{U}}$, $\det P = \frac{1}{2}\, g(P, P) \neq 0$. Then

$$(\det \Phi)\ \mathrm{Ad}[\Phi]v = \begin{pmatrix} 0 & (\det P)\, P\, \tilde{\bar{V}}\ \bar{P}^\star \\ (\det \bar{P}) \tilde{\bar{P}}\, V\ \ \tilde{P}^\star & 0 \end{pmatrix} \ .$$

By the same argument as before, $\Phi \in \mathrm{Cl}$ iff $\det P = \det \bar{P}$. □

Now it is not difficult to show that any complex 2×2-matrix with real determinant can be written as a product of Hermitian matrices. Using this, one recovers a well-known result:

Proposition 5.2 Cl *is multiplicatively generated by* $\boldsymbol{H}^\bullet \subset \boldsymbol{H}$*, the subset of all elements in* $\boldsymbol{H}$ *with non-vanishing Lorentz pseudo-norm.*

Namely any element of Cl can be written as

$$\Phi = v_1\, v_2\, \ldots\, v_n \ , \quad v_j \in \boldsymbol{H},\ g(v_j, v_j) \neq 0 \ ;$$

its inverse is

$$\Phi^{-1} = \frac{1}{\nu(\Phi)}\, v_n\, \ldots\, v_2\, v_1 \ , \quad \nu(\Phi) := g(v_1, v_1)\, g(v_2, v_2)\, \ldots\, g(v_n, v_n) \ .$$

Setting now $V_i \equiv \sqrt{2}\, v_i$ one has $\det V_i = \det \tilde{\bar{V}}_i = g(v_i, v_i)$, hence

$$\nu(\Phi) = \det\bigl(V_1\, \tilde{\bar{V}}_2\, V_3\, \tilde{\bar{V}}_4\, \ldots\bigr) = \prod_{i=1}^{n} \det(V_i) \ .$$

Namely, if $\Phi = \left(\begin{smallmatrix} K & 0 \\ 0 & K^\ddagger \end{smallmatrix}\right) \in \mathrm{Cl}^{(+)}$ then $\nu(\Phi) = \det K = \det K^\ddagger$; if $\Phi = \left(\begin{smallmatrix} 0 & P \\ P^\ddagger & 0 \end{smallmatrix}\right) \in \mathrm{Cl}^{(-)}$ then $\nu(\Phi) = \det P = \det P^\ddagger$.

Remark. Actually, it can be seen that any complex 2×2-matrix with real determinant can be written as a product of just *three* Hermitian matrices (but not, in general, of two matrices). This implies that an element in $\mathrm{Cl}^{(-)}$ can be written as $\left(\begin{smallmatrix} 0 & P \\ P^\ddagger & 0 \end{smallmatrix}\right)$ with $P = V_1\, V_2^\ddagger\, V_3$, and an element in $\mathrm{Cl}^{(+)}$ can be written as $\left(\begin{smallmatrix} K & 0 \\ 0 & K^\ddagger \end{smallmatrix}\right)$ with $K = V_1\, V_2^\ddagger\, V_3\, V_4^\ddagger$, $V_i \in \boldsymbol{H}^\bullet$.

The adjoint action of any $w \in \boldsymbol{H}$ on $\boldsymbol{H}$ is easily checked to be the negative of the reflection through the hyperplane orthogonal to w. It follows that $\mathrm{Cl}^{(+)}$ is the subgroup of all elements in Cl whose adjoint action preserves the orientation of $\boldsymbol{H}$. Moreover, the subgroup

$$\mathrm{Cl}^{\uparrow} := \{\Phi \in \mathrm{Cl} : \nu(\Phi) > 0\}$$

is constituted of all elements of Cl whose adjoint action preserves the time-orientation of $\boldsymbol{H}$. Its representation as $\Phi = v_1\, v_2\, \dots\, v_n$ has an even number of spacelike factors and any number of timelike factors.

The unit element of Cl is $1\!\!1 \in \boldsymbol{D}^{(+)} \subset \boldsymbol{D}$. Thus the Lie algebra of Cl is a 7-dimensional vector subspace

$$\mathfrak{L}\mathrm{Cl} \subset \boldsymbol{D}^{(+)} = \mathbb{R} \oplus \wedge^2\boldsymbol{H} \oplus \wedge^4\boldsymbol{H} \equiv \mathbb{R}\,1\!\!1 \oplus \hat{\gamma}(\wedge^2\boldsymbol{H}) \oplus \hat{\gamma}(\wedge^4\boldsymbol{H})\ .$$

Now observe that $\wedge^4\boldsymbol{H}$ is not contained in $\mathfrak{L}\mathrm{Cl}$ since

$$t \in \mathbb{R} \quad \Rightarrow \quad \exp(t\,\eta^{\#}) = \exp\begin{pmatrix} -\mathrm{i}\,t\,1\!\!1_{\boldsymbol{U}} & 0 \\ 0 & \mathrm{i}\,t\,1\!\!1_{\overline{\boldsymbol{U}}^\star} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\,t}\,1\!\!1_{\boldsymbol{U}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,t}\,1\!\!1_{\overline{\boldsymbol{U}}^\star} \end{pmatrix}$$

is not in Cl because the two component endomorphsims $\mathrm{e}^{-\mathrm{i}\,t}\,1\!\!1_{\boldsymbol{U}} \in \boldsymbol{U} \otimes \boldsymbol{U}^\star$ and $\mathrm{e}^{\mathrm{i}\,t}\,1\!\!1_{\overline{\boldsymbol{U}}^\star} \in \overline{\boldsymbol{U}}^\star \otimes \overline{\boldsymbol{U}}$ have non-real determinant. Hence, just by a dimension argument, one finds

$$\mathfrak{L}\mathrm{Cl} = \mathbb{R} \oplus \wedge^2\boldsymbol{H}\ .$$

5.2 Pin and Spin

If $\Phi \in \mathrm{Cl}$ and $a \in \mathbb{R} \setminus \{0\}$ then $\mathrm{Ad}[a\,\Phi] = \mathrm{Ad}[\Phi] : \boldsymbol{H} \to \boldsymbol{H}$. It is then natural to consider the subgroup

$$\mathrm{Pin} := \{\Phi \in \mathrm{Cl} : \nu(\Phi) = \pm 1\}\ ,$$

which is multiplicatively generated by all elements in $\boldsymbol{H}$ whose Lorentz pseudo-norm is ± 1. It has the subgroups

$$\mathrm{Spin} := \mathrm{Pin}^{(+)} \equiv \mathrm{Pin} \cap \mathrm{Cl}^{(+)} = \{\Phi \in \mathrm{Cl}^{(+)} : \nu(\Phi) = \pm 1\}\ ,$$

$$\mathrm{Pin}^{\uparrow} := \mathrm{Pin} \cap \mathrm{Cl}^{\uparrow} = \{\Phi \in \mathrm{Cl} : \nu(\Phi) = 1\}\ ,$$

$$\mathrm{Spin}^{\uparrow} := \mathrm{Spin} \cap \mathrm{Cl}^{\uparrow} = \{\Phi \in \mathrm{Cl}^{(+)} : \nu(\Phi) = 1\}\ .$$

These share the same Lie algebra

$$\wedge^2\boldsymbol{H} = \mathfrak{L}\mathrm{Pin} = \mathfrak{L}\mathrm{Spin} = \mathfrak{L}\mathrm{Pin}^{\uparrow} = \mathfrak{L}\mathrm{Spin}^{\uparrow}\ .$$

The automorphisms of $\boldsymbol{U}$ which have unit determinant constitute the group $\mathrm{Sl} \equiv \mathrm{Sl}(\boldsymbol{U})$; thus

$$\mathrm{Cl}^{(+)\uparrow} \equiv \mathrm{Cl}^{(+)} \cap \mathrm{Cl}^{\uparrow} = \left\{ \left(\begin{smallmatrix} K & 0 \\ 0 & K^{\ddagger} \end{smallmatrix}\right) \in \mathrm{End}\,\boldsymbol{W} : K \in \mathbb{R}^+ \times \mathrm{Sl} \right\}\ ,$$

$$\mathrm{Spin}^{\uparrow} = \left\{ \left(\begin{smallmatrix} K & 0 \\ 0 & K^{\ddagger} \end{smallmatrix}\right) \in \mathrm{End}\,\boldsymbol{W} : K \in \mathrm{Sl} \right\}\ .$$

In particular, one has the isomorphism

$$\mathrm{Spin}^{\uparrow} \leftrightarrow \mathrm{Sl} : \left(\begin{smallmatrix} K & 0 \\ 0 & K^{\ddagger} \end{smallmatrix}\right) \leftrightarrow K \ .$$

Now remember that

$$\hat{\gamma}(\wedge^2 \boldsymbol{H}) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A^{\ddagger} \end{smallmatrix}\right) \in \mathrm{End}\, \boldsymbol{W} : \mathrm{Tr}\, A = 0 \right\} \ ,$$

$$\hat{\gamma}(\mathbb{R} \oplus \wedge^2 \boldsymbol{H}) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A^{\ddagger} \end{smallmatrix}\right) \in \mathrm{End}\, \boldsymbol{W} : \Im\, \mathrm{Tr}\, A = 0 \right\} \ ;$$

moreover $\mathrm{End}\, \boldsymbol{U}$ can be decomposed into the direct sum of the subspace of all traceless endomorphism, which is just $\mathfrak{L}\mathrm{Sl}$, and the subspace $\mathbb{C}\, 1\!\!1$ generated by the identity. Then one has the Lie algebra isomorphisms

$$\mathfrak{L}\mathrm{Cl} = \mathfrak{L}\mathrm{Cl}^{(+)\uparrow} = \mathbb{R} \oplus \wedge^2 \boldsymbol{H} \longrightarrow (\mathbb{R}\, 1\!\!1) \oplus \mathfrak{L}\mathrm{Sl} \ ,$$

$$\mathfrak{L}\mathrm{Pin} = \mathfrak{L}\mathrm{Spin}^{\uparrow} = \wedge^2 \boldsymbol{H} \longrightarrow \mathfrak{L}\mathrm{Sl} \ .$$

Proposition 5.3 *Let*

$$\Phi = \left(\begin{smallmatrix} K & 0 \\ 0 & K^{\ddagger} \end{smallmatrix}\right) \in \mathrm{Spin} \ , \quad v \in \boldsymbol{H} \ , \quad \gamma(v) = \left(\begin{smallmatrix} V & 0 \\ 0 & V^{\ddagger} \end{smallmatrix}\right) \equiv \begin{pmatrix} \sqrt{2}\, v & 0 \\ 0 & \sqrt{2}\, v^{\ddagger} \end{pmatrix} \ .$$

Then

$$\mathrm{Ad}[\Phi]\gamma(v) = \pm \begin{pmatrix} 0 & [K{\otimes}\bar{K}](V) \\ \big([K{\otimes}\bar{K}](V)\big)^{\ddagger} & 0 \end{pmatrix} \ ,$$

where the $+$ *sign holds iff* $\Phi \in \mathrm{Spin}^{\uparrow}$*.*

PROOF: Remembering the previous results one finds

$$\mathrm{Ad}[\Phi]\gamma(v) = \frac{1}{\det K} \begin{pmatrix} 0 & K\, V\, \bar{K}^{\star} \\ \big(K\, V\, \bar{K}^{\star}\big)^{\ddagger} & 0 \end{pmatrix} \ .$$

Moreover

$$(K\, V\, \bar{K}^{\star})^{A\dot{A}} = K^{A}{}_{B}\, V^{B\dot{B}}\, (\bar{K}^{\star})_{\dot{B}}{}^{\dot{A}} = K^{A}{}_{B}\, V^{B\dot{B}}\, \bar{K}^{\dot{A}}{}_{\dot{B}} = (K \otimes \bar{K})^{A\dot{A}}{}_{B\dot{B}}\, V^{B\dot{B}} \ .$$

□

Now remember (§1.8) that the group $\{K \otimes \bar{K} \,:\, K \in \mathrm{Aut}(\boldsymbol{U})\}$ is constituted of automorphisms of $\boldsymbol{U} \otimes \overline{\boldsymbol{U}}$ which preserve the splitting $\boldsymbol{U} \otimes \overline{\boldsymbol{U}} = \boldsymbol{H} \oplus \mathrm{i}\, \boldsymbol{H}$ and the causal structure of $\boldsymbol{H}$. Its subgroup $\{K \otimes \bar{K} \,:\, K \in \mathrm{Sl}(\boldsymbol{U})\}$ coincides with $\mathrm{Lor}^{\uparrow}_{+}(\boldsymbol{H})$. Thus one sees that the group isomorphism $\mathrm{Sl} \to \mathrm{Spin}^{\uparrow}$ determines the 2-to-1 epimorphism $\mathrm{Spin}^{\uparrow} \to \mathrm{Lor}^{\uparrow}_{+}$.

One also finds that $\mathrm{Spin}^{\uparrow}$ is the subgroup of $\mathrm{End}\, \boldsymbol{W}$ preserving $(\gamma, \mathrm{k}, g, \eta, \varepsilon)$ as well as time-orientation. Let's review these properties in terms of two-spinors.

• Obviously, $\mathrm{Spin}^{\uparrow}$ preserves the splitting $\boldsymbol{W}=\boldsymbol{U}\oplus\overline{\boldsymbol{U}}^{\star}$. If $\Phi=\left(\begin{smallmatrix}K&0\\0&K^{\ddagger}\end{smallmatrix}\right)$, $K\in\mathrm{Sl}(\boldsymbol{U})$, then $\tilde{K}=K^{-1}$, so for $\psi\equiv(u,\chi),\psi'\equiv(u',\chi')\in\boldsymbol{W}$ one gets

$$\mathrm{k}(\Phi\psi,\Phi\psi')=\mathrm{k}\big((K\,u,\chi\,\bar{K}^{-1}),(K\,u',\chi'\,\bar{K}^{-1}\big)=\langle\bar{\chi}\,K^{-1},K\,u'\rangle+\langle\chi'\,\bar{K}^{-1},\bar{K}\,\bar{u}\rangle=$$
$$=\langle\bar{\chi},u'\rangle+\langle\chi',\bar{u}\rangle=\mathrm{k}(\psi,\psi')\;.$$

• Since $K\otimes\bar{K}:\boldsymbol{U}\otimes\overline{\boldsymbol{U}}\to\boldsymbol{U}\otimes\overline{\boldsymbol{U}}$ sends Hermitian tensors to Hermitian tensors and anti-Hermitian tensors to anti-Hermitian tensors, it preserves the splitting $\boldsymbol{U}\otimes\overline{\boldsymbol{U}}=\boldsymbol{H}\oplus\mathrm{i}\,\boldsymbol{H}$. Also, remember that $K\otimes\bar{K}=\mathrm{Ad}[\Phi]$.

• $K\otimes\bar{K}=\mathrm{Ad}[\Phi]\in\mathrm{Lor}^{\uparrow}_{+}(\boldsymbol{H})$, the subgroup of the Lorentz group which preserves orientation and time-orientation.

• Φ preserves the Dirac map γ. In fact if $y\in\boldsymbol{H}$ then

$$\gamma[y]=\begin{pmatrix}0&\sqrt{2}\,y\\\sqrt{2}\,y^{\ddagger}&0\end{pmatrix}\;,\quad y^{\ddagger}\equiv\tilde{\bar{y}}=\tilde{y}^{\star}\;,$$

$$\mathrm{Ad}[\Phi]\gamma[y]=\begin{pmatrix}0&\sqrt{2}\,[K\otimes\bar{K}]\,y\\\sqrt{2}\,([K\otimes\bar{K}]\,y)^{\ddagger}&0\end{pmatrix}=\gamma\big[[K\otimes\bar{K}]\,y\big]\;.$$

• If $K\in\mathrm{Sl}$ then K preserves any simplectic form $\varepsilon\in\wedge^2\boldsymbol{U}^{\star}$. Hence $\Phi\equiv\left(\begin{smallmatrix}K&0\\0&K^{\ddagger}\end{smallmatrix}\right)\in\mathrm{Spin}^{\uparrow}$ preserves the corresponding simplectic form $(\varepsilon,\bar{\varepsilon}^{\#})\in\wedge^2\boldsymbol{W}^{\star}$ and charge conjugation.

6 Spinors and particle momenta

6.1 Particle momentum in two-spinor terms

It has already been observed (§1.4) that any future-pointing non-spacelike element in $\boldsymbol{H}$ can be written in the form

$$u\otimes\bar{u}+v\otimes\bar{v}\;,\quad u,v\in\boldsymbol{U}\;.$$

If u and v are not proportional to each other, that is $\varepsilon(u,v)\neq 0$, then the above expression is a timelike future-pointing vector; if $\varepsilon(u,v)\neq 0$, then it is a null vector. Future-pointing elements in $\boldsymbol{H}$ are a contravariant, “conformally invariant” version of *classical particle momenta* (translation to a scaled and/or covariant version, when needed, will be effortless).

Let $\boldsymbol{K}$ and $\boldsymbol{N}$ be the subsets of $\boldsymbol{H}$ constituted of all future-pointing timelike vectors and of all future-pointing null vectors, respectively; moreover, set $\boldsymbol{J}:=\boldsymbol{K}\cup\boldsymbol{N}$ (all these sets do not contain the zero element). Consider now the real quadratic maps

$$\tilde{\mathrm{p}}:\boldsymbol{U}\times\boldsymbol{U}\to\boldsymbol{J}:(u,v)\mapsto\tfrac{1}{\sqrt{2}}\,(u\otimes\bar{u}+v\otimes\bar{v})\;,$$

$$\mathrm{p}:\boldsymbol{W}\cong\boldsymbol{U}\times\overline{\boldsymbol{U}}^{\star}\to\boldsymbol{J}:(u,\chi)\mapsto\tfrac{1}{\sqrt{2}}\,(u\otimes\bar{u}+\bar{\chi}^{\#}\otimes\chi^{\#})\;.$$

When a normalized symplectic form $\varepsilon\in\wedge^2\boldsymbol{U}^{\star}$ is *fixed*, $\tilde{\mathrm{p}}$ and p are essentially the same objects, as one can represent a given element $\tfrac{1}{\sqrt{2}}\,(u\otimes\bar{u}+v\otimes\bar{v})$ of $\boldsymbol{J}$ by writing

$v \otimes \bar{v}$ as $(\bar{\chi} \otimes \chi)^{\#}$; here, $u, v \in \boldsymbol{U}$, $\chi \in \overline{\boldsymbol{U}}^{\star}$. In such case I'll set

$$v := -\bar{\chi}^{\#} \qquad \Longleftrightarrow \qquad \chi = \bar{v}^{\flat} \ ,$$

$$\Rightarrow \quad \langle \bar{\chi}, u \rangle = \langle v^{\flat}, u \rangle = \varepsilon(v, u) \ , \quad \langle \chi, \bar{u} \rangle = \langle \bar{v}^{\flat}, \bar{u} \rangle = \bar{\varepsilon}(\bar{v}, \bar{u}) \ .$$

If $p = \mathrm{p}(u, \chi) \equiv \tilde{\mathrm{p}}(u, v)$ then we'll use the shorthands

$$\mu^2 := g(p, p) = 2\, |\varepsilon(u, v)|^2 = 2\, |\langle \bar{\chi}, u \rangle|^2 \ ,$$

$$h := \frac{\sqrt{2}}{\mu}\, \bar{p}^{\flat} = \frac{1}{|\langle \bar{\chi}, u \rangle|}\, (\bar{u}^{\flat} \otimes u^{\flat} + \chi \otimes \bar{\chi}) \ .$$

Then, h can be seen as an ε-normalized Hermitian metric on $\boldsymbol{U}$.

Proposition 6.1 *Let $(u, \chi) \equiv (u, \bar{v}^{\flat}) \in \boldsymbol{W}$, $\langle \bar{\chi}, u \rangle \neq 0$; let $p \in \boldsymbol{K}$. Then, the following conditions are equivalent:*

i) $p = u \otimes \bar{u} + (\bar{\chi} \otimes \chi)^{\#}$,

ii) $\gamma[p](u, \chi) = \mu \left(\mathrm{e}^{-\mathrm{i}\theta} u, \mathrm{e}^{\mathrm{i}\theta}\, \chi \right)$, $\theta \in \mathbb{R}$,

iii) $\bar{h}^{\flat}(u) = \mathrm{e}^{\mathrm{i}\theta}\, \chi$,

iv) $h^{\#}(\chi) = \mathrm{e}^{-\mathrm{i}\theta}\, u$,

v) $h(\bar{u}, v) = 0$ *and* $|\langle \bar{\chi}, u \rangle| = h(\bar{u}, u)$,

v') $h(\bar{u}, v) = 0$ *and* $|\langle \bar{\chi}, u \rangle| = h(\bar{v}, v)$,

where μ and h are defined in terms of (u, χ) as above.

PROOF: By straightorwaed calculations one sees that condition **i** implies conditions **ii**, **iii**, **iv**, **v** and **v'**. Moreover:

(ii ⇔ iii) : It follows from $\gamma[\tau](u, \chi) = \frac{1}{\sqrt{2}}\, \gamma[\bar{h}^{\#}](u, \chi) = \left(h^{\#}(\chi), \bar{h}^{\flat}(u) \right)$.

(iii ⇔ iv) : If $\bar{h}^{\flat}(u) = \mathrm{e}^{\mathrm{i}\theta}\, \chi$ then $u = h^{\#}(\bar{h}^{\flat}(u)) = h^{\#}(\mathrm{e}^{\mathrm{i}\theta}\, \chi) = \mathrm{e}^{\mathrm{i}\theta}\, h^{\#}(\chi)$.
Similarly, if $h^{\#}(\chi) = \mathrm{e}^{-\mathrm{i}\theta}\, u$ then $\chi = \bar{h}^{\flat}(h^{\#}(\chi)) = \bar{h}^{\flat}(\mathrm{e}^{-\mathrm{i}\theta}\, u) = \mathrm{e}^{-\mathrm{i}\theta}\, \bar{h}^{\flat}(u)$.

(iv ⇒ v) : $h(\bar{u}, v) = \langle h^{\flat}(\bar{u}), -\bar{\chi}^{\#} \rangle = -\langle \mathrm{e}^{-\mathrm{i}\theta}\, \bar{\chi}, \bar{\chi}^{\#} \rangle = \mathrm{e}^{-\mathrm{i}\theta}\, \varepsilon^{\#}(\bar{\chi}, \bar{\chi}) = 0$.
Moreover $h(\bar{u}, u) = \langle \bar{h}^{\flat}(u), \bar{u} \rangle = \langle \mathrm{e}^{\mathrm{i}\theta}\, \chi, \bar{u} \rangle = \langle \bar{\chi}, u \rangle\, \langle \chi, \bar{u} \rangle / |\langle \bar{\chi}, u \rangle| = |\langle \bar{\chi}, u \rangle|$.

(v ⇒ iv) : From $0 = h(\bar{u}, v) = \langle h^{\flat}(\bar{u}), -\bar{\chi}^{\#} \rangle = -\varepsilon^{\#}(\bar{\chi}, h^{\flat}(\bar{u}))$ one has $\bar{\chi} = c\, h^{\flat}(\bar{u})$, $c \in \mathbb{C}$. Then $\langle \bar{\chi}, u \rangle = c\, h(\bar{u}, u) = c\, |\langle \bar{\chi}, u \rangle| \quad \Rightarrow \quad c = \mathrm{e}^{\mathrm{i}\theta}$.

(v ⇒ v') : From **iv** (equivalent to **v**) one has $h(\bar{v}, v) = \langle h, \chi^{\#} \otimes \bar{\chi}^{\#} \rangle = \langle h^{\#}, \chi \otimes \bar{\chi} \rangle = \langle h^{\#}(\chi), \bar{\chi} \rangle = \mathrm{e}^{-\mathrm{i}\theta}\, \langle \bar{\chi}, u \rangle = |\langle \bar{\chi}, u \rangle|$, hence also $h(\bar{v}, v) = |\langle \bar{\chi}, u \rangle|$.

(v' ⇒ iv) : As in **v** ⇒ **iv** one has $\bar{\chi} = c\, h^{\flat}(u)$, $c \in \mathbb{C}$, or $u = \frac{1}{c}\, h^{\#}(\chi)$. Then, from $\langle \bar{\chi}, u \rangle = \langle \bar{\chi}, \frac{1}{\bar{c}}\, h^{\#}(\chi) \rangle = \frac{1}{\bar{c}}\, h^{\#}(\chi, \bar{\chi}) = \frac{1}{\bar{c}}\, h(\bar{v}, v)$ one has $\bar{c} = \mathrm{e}^{-\mathrm{i}\theta}$ i.e. $c = \mathrm{e}^{\mathrm{i}\theta}$.

(v ⇒ i) : Using also **v'** (already seen to be equivalent to **v**) one sees that the couple $(\zeta_u\, , \zeta_v) \equiv (u, v)/\sqrt{|\langle \bar{\chi}, u \rangle|}$ is an h-orthonormal basis of $\boldsymbol{U}$; hence $h^{\#} = \bar{\zeta}_u \otimes \zeta_u + \bar{\zeta}_v \otimes \zeta_v = \frac{1}{|\langle \bar{\chi}, u \rangle|} \left(\bar{u} \otimes u + \bar{v} \otimes v \right)$. Condition **i** then follows. □

6.2 Bundle structure of 4-spinor space over momentum space

The previous results show that the restriction $\mathrm{p}:\boldsymbol{W}\setminus\{0\}\;\longrightarrow\;\boldsymbol{J}$ is surjective. Since the Lorentz "length" of $\mathrm{p}(u,\chi)$ is $\sqrt{2}\,|\langle\bar\chi,u\rangle|$ one sees that the subset of all elements in $\boldsymbol{W}$ which project onto $\boldsymbol{N}$ is the 6-dimensional real submanifold

$$\boldsymbol{W}^0:=\mathrm{p}^{-1}(\boldsymbol{N})=\big\{(u,\chi)\in\boldsymbol{W}\setminus\{0\}:\langle\bar\chi,u\rangle=0\big\}\subset\boldsymbol{W}\;.$$

The subset of all elements in $\boldsymbol{W}$ which project onto $\boldsymbol{K}$ is the open submanifold

$$\boldsymbol{W}^{\backprime}:=\mathrm{p}^{-1}(\boldsymbol{K})=\big\{(u,\chi)\in\boldsymbol{W}:\langle\bar\chi,u\rangle\neq0\big\}\;,$$

and one has

$$\boldsymbol{W}\setminus\{0\}=\boldsymbol{W}^0\cup\boldsymbol{W}^{\backprime}\;.$$

Moreover, consider the subsets $\boldsymbol{W}^+,\boldsymbol{W}^-\subset\boldsymbol{W}^{\backprime}$ defined to be

$$\boldsymbol{W}^{\pm}:=\big\{(u,\chi)\in\boldsymbol{W}:\langle\bar\chi,u\rangle\in\mathbb{R}^{\pm}\big\}\;.$$

Recalling condition **ii** of proposition 6.1 one has

$$\gamma[\mathrm{p}\psi]\psi=\mu\,(\mathrm{e}^{-\mathrm{i}\theta}u,\mathrm{e}^{\mathrm{i}\theta}\,\chi)\;,$$

which holds for every $\psi\equiv(u,\chi)\in\boldsymbol{W}$ (if $\psi\in\boldsymbol{W}^0$ then $\mu=0$). In particular

$$\boldsymbol{W}^{\pm}=\big\{\;\psi\equiv(u,\chi)\in\boldsymbol{W}\setminus\{0\}:\gamma[\mathrm{p}\psi]\psi=\pm\mu\,\psi\;,\mu\equiv|\langle\bar\chi,u\rangle|\;\big\}\;.$$

Next, consider the subset

$$\tilde{\boldsymbol{W}}^{\backprime}:=\{(u,v):\varepsilon(u,v)\neq0\}\subset\boldsymbol{U}\times\boldsymbol{U}\;,$$

and note that when a normalized symplectic form $\varepsilon\in\wedge^2\boldsymbol{U}^\star$ is fixed, $\tilde{\boldsymbol{W}}^{\backprime}$ can be identified with $\boldsymbol{W}^{\backprime}$ via the correspondence $\bar v^\flat\leftrightarrow\chi$. $\tilde{\boldsymbol{W}}^{\backprime}$ is a fibred set over $\boldsymbol{K}$; for each $p\in\boldsymbol{K}$, the fibre of $\tilde{\boldsymbol{W}}^{\backprime}$ over p is the subset

$$\tilde{\boldsymbol{W}}^{\backprime}_p:=\tilde{\mathrm{p}}^{-1}(p)=\big\{(u,v)\in\tilde{\boldsymbol{W}}^{\backprime}:\tfrac{1}{\sqrt2}\,(u\otimes\bar u+v\otimes\bar v)=p\big\}\;.$$

Proposition 6.2 $\tilde{\mathrm{p}}:\tilde{\boldsymbol{W}}^{\backprime}\to\boldsymbol{K}$ *is a trivializable principal bundle with structure group* $\mathrm{U}(2)$ *.*

PROOF: Let $p=\tilde{\mathrm{p}}(u,v)=\tilde{\mathrm{p}}(u',v')$. From proposition 6.1 one then sees that (u,v) and (u',v') are orthonormal bases of $\boldsymbol{U}$ relatively to the Hermitian metric $h\equiv\sqrt2\,\bar p^\flat/\mu$. Then there exists a unique transformation $K\in\mathrm{U}(\boldsymbol{U},h)$ such that

$$u'=K(u)\;,\quad v'=K(v)\;;$$

hence, $\tilde{\boldsymbol{W}}^{\backprime}_p$ is a group-affine space, with derived group $\mathrm{U}(2)$.

Let now (ζ_A) be an ε-normalized basis of $\boldsymbol{U}$ and (τ_λ) the associated Pauli frame. For each $p\in\boldsymbol{K}$ let $L_p\in\mathrm{Lor}^{\uparrow}_{+}(\boldsymbol{H})$ be the boost such that $L_p\tau_0=p/\mu$, where $\mu^2\equiv g(p,p)$; up to sign there is a unique $B_p\in\mathrm{Sl}(\boldsymbol{U})$ such that $L_p=B_p\otimes\bar B_p$, and a consistent smooth way of choosing one such B_p for each p can be fixed. It turns out that the basis $\big(\sqrt{\mu}\,B_p\zeta_A\big)$ is orthonormal relatively to $\sqrt2\,\bar p^\flat/\mu$ seen as a Hermitian metric on $\boldsymbol{U}$, hence $\tilde{\mathrm{p}}(\sqrt{\mu}\,B_p\zeta_1\,,\sqrt{\mu}\,B_p\zeta_2)=p$. In this way one selects an "origin" element in each fibre of $\tilde{\mathrm{p}}$, so getting a trivialization $\tilde{\boldsymbol{W}}^{\backprime}\to\boldsymbol{K}\times\mathrm{U}(2)$. □

Using a little two-spinor algebra it is not difficult to prove:

Proposition 6.3 *Let* $\psi, \psi' \in \boldsymbol{W}^\backprime$, $\psi \equiv (u, \chi)$, $\psi' \equiv (u', \chi')$; *let* $K \in \operatorname{Aut}\boldsymbol{U}$ *be the unique automorphism of* $\boldsymbol{U}$ *such that*

$$K\,u = u\;, \quad K\,\bar{\chi}^{\#} = \bar{\chi}'^{\#}\;.$$

Then

$$K = \frac{1}{\langle\bar{\chi}, u\rangle^2}\left[\langle\bar{\chi}, u'\rangle\, u \otimes \bar{\chi} - \varepsilon^{\#}(\bar{\chi}, \bar{\chi}')\, u \otimes u^\flat + \varepsilon(u, u')\, \bar{\chi}^{\#} \otimes \bar{\chi} + \langle\bar{\chi}', u\rangle\, \bar{\chi}^{\#} \otimes u^\flat\right]\;.$$

Moreover, one has

$$\chi' = K^{\ddagger}\,\chi\;.$$

Conversely, the conditions $u' = Ku$ *and* $\chi' = K^{\ddagger}\chi$ *determine* K *uniquely.*

The above expression for K is invariant relatively to the transformation $\varepsilon \mapsto \mathrm{e}^{\mathrm{i}\,\theta}\varepsilon$; hence, K is independent of the particular normalized symplectic form ε chosen.

When a normalized $\varepsilon \in \wedge^2\boldsymbol{U}^\star$ is given, one has the real vector bundle isomorphism $\boldsymbol{W}^\backprime \leftrightarrow \tilde{\boldsymbol{W}}^\backprime : (u, v) \leftrightarrow (u, \bar{v}^\flat)$. Through this correspondence, $\boldsymbol{W}^\backprime \to \boldsymbol{K}$ turns out to be a trivializable principal bundle with structure group $\mathrm{U}(2)$. If $\psi, \psi' \in \boldsymbol{W}^\backprime_p$, let

$$(K) = c\begin{pmatrix} a & \bar{b} \\ -b & \bar{a}\end{pmatrix} \in \mathrm{U}(2)\;, \quad a, b, c \in \mathbb{C} : |a|^2 + |b|^2 = |c|^2 = 1\;,$$

be the matrix of $K \in \operatorname{Aut}\boldsymbol{U}$ sending ψ to ψ' (according to proposition 6.3) relatively to the basis (u, v). Then

$$\begin{cases} u' = c\,(a\,u - b\,v)\;, \\ v' = c\,(\bar{b}\,u + \bar{a}\,v)\;, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} u' = c\,(a\,u + b\,\bar{\chi}^{\#})\;, \\ \chi' = \bar{c}\,(a\,\chi + b\,\bar{u}^\flat)\;. \end{cases}$$

If you take a different normalized symplectic form $\varepsilon \to \mathrm{e}^{\mathrm{i}\,\theta}\varepsilon$, then K does not change, while the corresponding matrix $(K) \in \mathrm{U}(2)$ changes according to $c \to c$, $a \to a$, $b \to \mathrm{e}^{\mathrm{i}\,\theta} b$.

The above U(2)-action does not preserve $\boldsymbol{W}^{\pm} \subset \boldsymbol{W}^\backprime$. In fact it's straightforward to prove:

Proposition 6.4 *Let* $\psi, \psi' \in \boldsymbol{W}^+_p$ *(resp.* $\psi, \psi' \in \boldsymbol{W}^-_p$*),* $\psi \equiv (u, \chi)$, $\psi' \equiv (u', \chi')$; *let* K *be the unique automorphism of* $\boldsymbol{U}$ *such that* $Ku = u$, $K^{\ddagger}\chi = \chi'$. *Then* $K \in \mathrm{SU}(\boldsymbol{U}, h)$, *where* $h \equiv \sqrt{2}\,\bar{p}^\flat/\mu$.

Hence, $\boldsymbol{W}^+ \to \boldsymbol{K}$ and $\boldsymbol{W}^- \to \boldsymbol{K}$ turn out to be trivializable principal bundles, with structure group $\mathrm{SU}(2)$.

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